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Extensions of a theorem of Hsu and Robbins

on the convergence rates in the law of large numbers

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1 Introduction

1.1 Convergence rate in the law of large numbers: the iid case

Consider i.i.d. r.v. X_i with $EX_i = 0$. Let

$$S_n = X_1 + ... + X_n.$$

Law of Large numbers:

$$\frac{S_n}{n} \to 0.$$

Question: at what rate $P(|S_n| > n\varepsilon) \to 0$?

The theorem of Hsu-Robbins-Erdos

Hsu and Robbins (1947):

$$EX_1^2 < \infty \Rightarrow \sum_n P(|S_n| > n\varepsilon) < \infty \quad \forall \varepsilon > 0.$$

(”complete convergence”, which implies a.s. convergence)

Erdos (1949): the converse also holds:

$$EX_1^2 < \infty \Leftarrow \sum_n P(|S_n| > n\varepsilon) < \infty \quad \forall \varepsilon > 0.$$

Spitzer (1956):

$$\sum_n n^{-1} P(|S_n| > n\varepsilon) < \infty \quad \forall \varepsilon > 0 \text{ whenever } EX_1 = 0.$$

Baum and Katz (1965): for $p > 1$,

$$\boldsymbol{E}|\boldsymbol{X}_1|^p < \infty \Leftrightarrow \sum_n n^{p-2} P(|S_n| > n\varepsilon) < \infty \quad \forall \varepsilon > 0;$$

in particular,

$$\boldsymbol{E}|\boldsymbol{X}_1|^p < \infty \Rightarrow P(|S_n| > n\varepsilon) = o(n^{-(p-1)})$$

Question: is it valid for martingale differences?

1.2 Convergence rates in the law of large numbers: the martingale case

Is the theorem of Baum and Katz (1965) still valid for martingale differences (X_j)?

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset ...,$$

$\forall j$, X_j are $\mathcal{F}_j$ measurable with $E[X_j|\mathcal{F}_{j-1}] = 0$

$$(\Leftrightarrow S_n = X_1 + ... + X_n \text{ is a martingale. })$$

Lesigne and Volney (2001): $p \geq 2$

$$E|X_1|^p < \infty \Rightarrow P(|S_n| > n\varepsilon) = o(n^{-p/2})$$

and the exponent $p/2$ is the best possible, even for stationary and ergodic sequences of martingale differences.

Therefore the theorem of Baum and Katz does not hold for martingale differences without additional conditions.

[Curiously, Stoica (2007) claimed that the theorem of Baum and Katz still holds for $p > 2$ in the case of martingale differences without additional assumption. His claim is a contradiction with the conclusion of Lesigne and Volney (2001), and his proof is wrong: he chose an element in an empty set!]

1.3 Under what conditions the theorem of Baum and Katz still holds for martingale differences?

Alsmeyer (1990) proved that the theorem of Baum and Katz of order $p > 1$ still holds for martingale differences (X_j) if for some $\gamma \in (1, 2]$ and $q > (p-1)/(\gamma-1)$,

$$\sup_{n \geq} \|\frac{1}{n} \sum_{j=1}^{n} E[|X_j|^\gamma | \mathcal{F}_{j-1}]\|_q < \infty$$

where $\|.\|_q$ denotes the L^q norm.
His result is already nice, but:

(a) it does not apply to ”non-homogeneous cases”, such as martingales of the form

$$S_n = \sum_{j=1}^{n} j^a X_j,$$

where $a > 0$, X_j are identically distributed;
(b) in applications (e.g. in the study of directed polymers in a random environment), instead of a single martingale, we need to consider martingale arrays:

$$S_{n,\infty} = \sum_{j=1}^{\infty} X_{n,j},$$

where for each n, $\{X_{n,j} : j \geq 1\}$ are martingale differences with respect to some filtration $\{\mathcal{F}_{n,j} : j \geq 0\}$.

Our objective: extend the theorem of Baum and Katz (1965) to a large class of martingale arrays, in improving Alsmeyer's result for martingales, by establishing a sharp comparison result between

$$P(\sum_{j=1}^{\infty} X_{n,j} > \varepsilon) \text{ and } \sum_{j=1}^{\infty} P(X_{n,j} > \varepsilon)$$

for arrays of martingale differences $\{X_{n,j} : j \geq 1\}$.

Our result is sharper then the known ones even in the independent (not necessarily identically distributed) case.

2. Main results for martingale arrays

For $n \geq 1$, let $\{(X_{nj}, \mathcal{F}_{nj}) : j \geq 1\}$ be a sequence of martingale differences, and write

$$m_n(\gamma) = \sum_{j=1}^{\infty} \mathbb{E}[|X_{nj}|^{\gamma}|\mathcal{F}_{n,j-1}], \quad \gamma \in (1, 2],$$

$$S_{n,j} = \sum_{i=1}^{j} X_{ni}, \quad j \geq 1,$$

$$S_{n,\infty} = \sum_{i=1}^{\infty} X_{ni}.$$

Lemma 1 (Law of large numbers) If for some $\gamma \in (1, 2]$,

$$\mathbb{E}m_n(\gamma) := \sum_{j=1}^{\infty} \mathbb{E}[|X_{nj}|^{\gamma}] \to 0,$$

then for all $\varepsilon > 0$,

$$P\{\sup_{j \geq 1} |S_{n,j}| > \varepsilon\} \to 0$$

and

$$P\{|S_{n,\infty}| > \varepsilon\} \to 0.$$

We are interested in their convergence rates.

Theorem 1 Let $\Phi : \mathbb{N} \mapsto [0, \infty)$. Suppose that for some $\gamma \in (1, 2]$, $q \in [1, \infty)$ and $\varepsilon_0 \in (0, 1)$,

$$\mathbb{E}m_n^q(\gamma) \to 0 \text{ and } \sum_{n=1}^{\infty} \Phi(n)(\mathbb{E}m_n^q(\gamma))^{1-\varepsilon_0} < \infty. \qquad (C1)$$

Then the following assertions are all equivalent:

$$\sum_{n=1}^{\infty} \Phi(n) \sum_{j=1}^{\infty} P\{|X_{nj}| > \varepsilon\} < \infty \ \forall \varepsilon > 0; \qquad (1)$$

$$\sum_{n=1}^{\infty} \Phi(n) P\{\sup_{j \geq 1} |S_{nj}| > \varepsilon\} < \infty \ \forall \varepsilon > 0; \qquad (2)$$

$$\sum_{n=1}^{\infty} \Phi(n) P\{|S_{n,\infty}| > \varepsilon\} < \infty \ \forall \varepsilon > 0. \qquad (3)$$

Remark. The condition (C1) holds if for some $r \in \mathbb{R}$ and $\varepsilon_1 > 0$,

$$\Phi(n) = O(n^r) \text{ and } \|m_n(\gamma)\|_\infty = O(n^{-\varepsilon_1}). \qquad (C1')$$

In the case where this holds with $\gamma = 2$, Ghosal and Chandra (1998) proved that (1) implies (2); our result is sharper because we have the equivalence.

Theorem 2 Let $\Phi : \mathbb{N} \mapsto [0, \infty)$ be such that $\Phi(n) \to \infty$. Suppose that for some $\gamma \in (1, 2]$, $q \in [1, \infty)$ and $\varepsilon_0 \in (0, 1)$,

$$\Phi(n)(\mathbb{E}m_n^q(\gamma))^{1-\varepsilon_0} = o(1) \quad (resp. O(1)). \tag{C2}$$

Then the following assertions are all equivalent:

$$\Phi(n) \sum_{j=1}^{\infty} P\{|X_{nj}| > \varepsilon\} = o(1) \quad (resp. O(1)) \quad \forall \varepsilon > 0; \tag{4}$$

$$\Phi(n) P\{\sup_{j \geq 1} |S_{nj}| > \varepsilon\} = o(1) \quad (resp. O(1)) \quad \forall \varepsilon > 0; \tag{5}$$

$$\Phi(n) P\{|S_{n,\infty}| > \varepsilon\} = o(1) \quad (resp. O(1)) \quad \forall \varepsilon > 0. \tag{6}$$

3. Consequences for martingales We now consider the single martingale case

$$S_j = X_1 + ... + X_j$$

w.r.t. a filtration

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset ...$$

By definition, $E[X_j | \mathcal{F}_{j-1}] = 0$.

For simplicity, let us only consider the case where

$$\Phi(n) = n^{p-2} \ell(n),$$

where $p > 1$, ℓ is a function slowly varying at ∞:

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \quad \forall \lambda > 0.$$

Notice that

$$S_n/n \to 0 \text{ a.s. iff } P(\sup_{j\geq n} \frac{|S_j|}{j} > \varepsilon) \to 0 \forall \varepsilon > 0.$$

To consider its rate of convergence, we shall use the condition that for some $\gamma \in (1, 2]$ and $q \in [1, \infty)$ with $q > (p-1)/(\gamma-1)$,

$$\sup_{n\geq 1} \|\underline{m}_n(\gamma, n)\|_q < \infty, \qquad (C3)$$

where $\underline{m}_n(\gamma, n) = \frac{1}{n}\sum_{j=1}^n \mathbb{E}[|X_j|^\gamma|\mathcal{F}_{j-1}]$. Remark that (C3) holds evidently if for some constant $C > 0$ and all $j \geq 1$,

$$\mathbb{E}[|X_j|^\gamma|\mathcal{F}_{j-1}] \leq C \quad a.s. \qquad (C4)$$

Theorem 3 Let $p > 1$ and $\ell \geq 0$ be slowly varying at ∞. Under (C3) or (C4), the following assertions are equivalent:

$$\sum_{n=1}^{\infty} n^{p-2}\ell(n) \sum_{j=1}^{n} P\{|X_j| > n\varepsilon\} < \infty \quad \forall \varepsilon > 0; \tag{7}$$

$$\sum_{n=1}^{\infty} n^{p-2}\ell(n) P\{\sup_{1\leq j\leq n} |S_j| > n\varepsilon\} < \infty \quad \forall \varepsilon > 0; \tag{8}$$

$$\sum_{n=1}^{\infty} n^{p-2}\ell(n) P\{|S_n| > n\varepsilon\} < \infty \quad \forall \varepsilon > 0. \tag{9}$$

$$\sum_{n=1}^{\infty} n^{p-2}\ell(n) P\{\sup_{j\geq n} \frac{|S_j|}{j} > \varepsilon\} < \infty \quad \forall \varepsilon > 0. \tag{10}$$

Remark. If X_j are identically distributed, then (7) is equivalent to the moment condition

$$E|X_1|^p \ell(|X_1|) < \infty.$$

So Theorem 3 is an extension of the result of Baum and Katz (1965). When ℓ is a constant, it was proved by Alsmeyer (1991).

Theorem 4 Let $p > 1$ and $\ell \geq 0$ be slowly varying at ∞. Under (C3) or (C4), the following assertions are equivalent:

$$n^{p-1}\ell(n)\sum_{j=1}^{n} P\{|X_j| > n\varepsilon\} = o(1) \quad (resp.\ O(1)) \quad \forall\varepsilon > 0; \tag{11}$$

$$n^{p-1}\ell(n)P\{\sup_{1\leq j\leq n} |S_j| > n\varepsilon\} = o(1) \quad (resp.\ O(1)) \quad \forall\varepsilon > 0; \tag{12}$$

$$n^{p-1}\ell(n)P\{|S_n| > n\varepsilon\} = o(1) \quad (resp.\ O(1)) \quad \forall\varepsilon > 0. \tag{13}$$

$$n^{p-1}\ell(n)P\{\sup_{j\geq n} \frac{|S_j|}{j} > \varepsilon\} = o(1) \quad (resp.\ O(1)) \quad \forall\varepsilon > 0. \tag{14}$$

4. Applications to sums of weighted random variables.

Example: Ces$\grave{a}$ro summation for martingale differences.

For $a > -1$, let $A_0^a = 1$ and

$$A_n^a = \frac{(\alpha+1)(a+2)\cdots(a+n)}{n!}, \quad n \geq 1.$$

Then $A_n^a \sim \frac{n^a}{\Gamma(a+1)}\ as\ n \to \infty$, and $\frac{1}{A_n^a}\sum_{j=0}^n A_{n-j}^{a-1} = 1.$ We consider convergence rates of

$$\frac{\sum_{j=0}^n A_{n-j}^{a-1} X_j}{A_n^a},$$

where $\{(X_j, \mathcal{F}_j), j \geq 0\}$ are martingale differences that are identically distributed.

For simplicity, suppose that for some $\gamma \in (1,2], C > 0$ and all $j \geq 1$,

$$\mathbb{E}\left[|X_j|^\gamma \middle| \mathcal{F}_{j-1}\right] \leq C \ a.s. \tag{15}$$

Theorem 5. Let $\{(X_j, \mathcal{F}_j), j \geq 0\}$ be identically distributed martingale differences satisfying (15). Let $p \geq 1$, and assume that

$$\begin{cases} \mathbb{E}|X_1|^{\frac{p-1}{a+1}} < \infty & \text{if } 0 < a < 1 - \frac{1}{p}, \\ \mathbb{E}|X_1|^p \log(e \vee |X_1|) < \infty & \text{if } a = 1 - \frac{1}{p}, \\ \mathbb{E}|X_1|^p < \infty & \text{if } 1 - \frac{1}{p} < a \leq 1. \end{cases} \tag{16}$$

Then

$$\sum_{n=1}^{\infty} n^{p-2} P\{|\sum_{j=0}^{n} A_{n-j}^{a-1} X_j| > A_n^a \varepsilon\} < \infty \text{ for all } \varepsilon > 0. \tag{17}$$

Remark: in the independent case, the result is due to Gut (1993).

5. Proofs of main results

The proofs are based on some maximal inequalities for martingales.

A. Relation between

$$P(\max_{1 \le j \le n} |X_j| > \varepsilon) \text{ and } P(\max_{1 \le j \le n} |S_j| > \varepsilon)$$

for martingale differences (X_j):

Lemma A Let $\{(X_j, \mathcal{F}_j), 1 \le j \le n\}$ be a finite sequence of martingale differences. Then for any $\varepsilon > 0, \gamma \in (1, 2], q \ge 1$, and $L \in \mathbb{N}$,

$$P\{\max_{1\le j\le n} |X_j| > 2\varepsilon\} \le P\{\max_{1\le j\le n} |S_j| > \varepsilon\}$$

$$\le P\{\max_{1\le j\le n} |X_j| > \frac{\varepsilon}{4(L+1)}\}$$

$$+C\varepsilon^{\frac{-q\gamma(L+1)}{q+L}} (\mathbb{E} m_n^q(\gamma))^{\frac{1+L}{q+L}}, \quad (18)$$

where $C = C(\gamma, q, L) > 0$ is a constant depending only on γ, q and L,

$$m_n(\gamma) = \sum_{j=1}^{n} \mathbb{E}[|X_j|^\gamma | \mathcal{F}_{j-1}].$$

B. Relation between

$$P(\max_{1\le j\le n} X_j > \varepsilon) \text{ and } \sum_{1\le j\le n} P(X_j > \varepsilon)$$

for adapted sequences (X_j):

Lemma B Let $\{(X_j, \mathcal{F}_j), 1 \le j \le n\}$ be an adapted sequence of r.v. Then for $\varepsilon > 0, \gamma > 0$ and $q \ge 1$,

$$P\{\max_{1\le j\le n} X_j > \varepsilon\} \le \sum_{j=1}^{n} P\{X_j > \varepsilon\}$$

$$\le (1+\varepsilon^{-\gamma})P\{\max_{1\le j\le n} X_j > \varepsilon\} + \varepsilon^{-\gamma}\mathbb{E}m_n^q(\gamma),$$

where $m_n(\gamma) = \sum_{j=1}^{n} \mathbb{E}[|X_j|^\gamma|\mathcal{F}_{j-1}]$.

C. Relation between

$$P(\max_{1 \leq j \leq n} |S_j| > \varepsilon) \text{ and } P(|S_n| > \varepsilon)$$

for martingale differences (X_j):

Lemma C Let $\{(X_j, \mathcal{F}_|), \infty \leq | \leq \backslash\}$ be a finite sequence of martingale differences. Then for $\varepsilon > 0, \gamma \in (1, 2]$ and $q \geq 1$,

$$P\{\max_{1 \leq j \leq n} |S_j| > \varepsilon\} \leq 2P\{|S_n| > \frac{\varepsilon}{2}\}$$
$$+\varepsilon^{-q\gamma} 2^{q(\gamma+1)} C^q(\gamma) \mathbb{E} m_n^q(\gamma),$$

where $m_n(\gamma) = \sum_{j=1}^{n} \mathbb{E}[|X_j|^\gamma | \mathcal{F}_{j-1}]$,
$C(\gamma) = \left(18\gamma^{3/2}/(\gamma - 1)^{1/2}\right)^\gamma$.

Thank you!

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