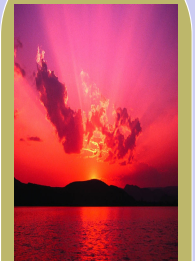

Strong Law of large number Law of the iterated logarithm for nonlinear probabilities

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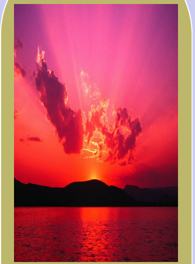
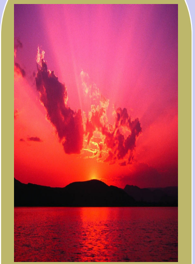


Outline

- ♢ ***History of LLN and LIL for probabilities***
- ♢ ***Why to study LLN and LIL for capacities***
- ♢ ***Nonlinear probabilities and nonlinear expectations***
- ♢ ***Main results***
- ♢ ***Applications***



0.1. History of LLN and LIL for probability

⋆ Law of large number(LLN):

(1) Brahmagupta (598-668), Cardano (1501-1576)

(2) Jakob Bernoulli(1713), Poisson (1835)

(3) Chebyshev, Markov, Borel(1909), Cantelli and Kolmogorov(IID).

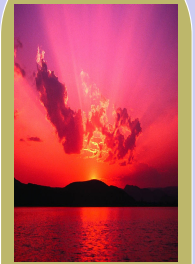
⋆ Law of iterated logarithm(LIL):

(1) Khintchine(1924) for Bernoulli model

Kolmogorov(1929), Hartman–Wintner(1941) (IID)

(2) Levy(1937) for Martingale

(3) Strassen(1964) for functional random variables.



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0.2. Strong LLN and LIL for probabilities

Assumption: $\{X_i\}$ IID , $S_n/n := \sum_{i=1}^n X_i$, $EX_1 = \mu$, Then

Theorem 1: Kolmogorov:

$$P(\lim_{n\to\infty} S_n/n = \mu) = 1$$

Theorem 2: Hartman–Wintner(1941): If $EX_1 = 0$, $EX_1^2 = \sigma^2$, Then

(a)

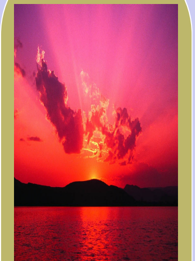
$$P\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = \sigma\right) = 1$$

(b)

$$P\left(\liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -\sigma\right) = 1$$

(c) Suppose that $C(\{x_n\})$ is the cluster set of a sequence of $\{x_n\}$ in R, then

$$P\left(C(\{\omega : S_n(\omega)/\sqrt{2n\mathrm{loglog}n}\}) = [-\sigma, \sigma]\right) = 1.$$

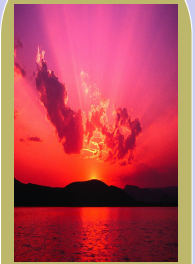


0.3. Why to study LLN and LIL in Finance

THEOREM 1 *(Black-Scholes, 1973:) In complete markets, there exists a unique probability measure Q, such that the pricing of option ξ at strike date T is given by $E_Q[\xi e^{-rT}]$. Where $r = 0$ is interest rate of bond.*

Monte Carlo, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = E_Q[\xi]$.

- ⋆ (Linear) expectation $\leftarrow$ Black-Scholes $\rightarrow$ Complete Markets
- ⋆ $\inf_{Q\in\mathcal{P}} E_Q[\xi], \sup_{Q\in\mathcal{P}} E_Q[\xi] \iff$ Incomplete Markets, Q is not unique, SET $\mathcal{P}$.
- ⋆ Super-pricing: $\inf_{Q\in\mathcal{P}} E_Q[\xi], \sup_{Q\in\mathcal{P}} E_Q[\xi]$. Nonlinear expectation! $\lim_{n\to\infty} S_n/n =?$



0.4. Bernoulli Trials with ambiguity

⋆ Bernoulli Trials:

Repeated independent trials are called Bernoulli trials if there are only two possible outcomes for each trial and their probabilities REMAIN (are no longer) the same throughout the trials.

⋆ Let $X_i = 1$ if head occurs and $X_i = 0$ if tail occurs.

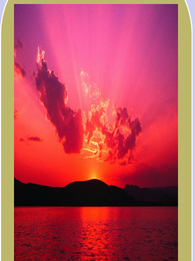
$P_\theta(X_i = 1) = \theta, \quad P_\theta(X_i = 0) = 1 - \theta, \quad S_n := \sum_{i=1}^n X_i$

⋆ If $\theta = 1/2$ (Unbalance), LLN stats

$$P_\theta(\lim_{n\to\infty} S_n/n = 1/2) = 1$$

Or

$$\lim_{n\to\infty} S_n/n = 1/2 \quad a.s \quad (P_\theta)$$



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⋆ If a coin is balance. $P_\theta(X_i = 1) = \theta \in [1/3, 1/2]$.

Let $\mathcal{P} := \{P_\theta, \theta \in [1/3, 1/2]\}$.

$E_{P_\theta}[X_i] = \theta$ Unknown,

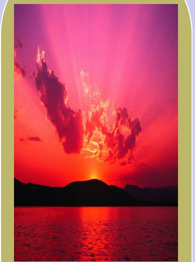
But $\max_{P\in\mathcal{P}} E_P[X_i] = 1/2, \quad \min_{P\in\mathcal{P}} E_P[X_i] = 1/3$.

⋆ **Question:** what is the limit $S_n/n \to ?$

(a) Capacity: If $V(A) := \max_{P\in\mathcal{P}} P(A), \quad v(A) := \min_{P\in\mathcal{P}} P(A)$

Can S_n/n converge to $\max_{P\in\mathcal{P}} E_P[X_i]$ or $\min_{P\in\mathcal{P}} E_P[X_i]$ a.s. V or v?

(b) The relation between the set of limit points of S_n/n and the interval of $\min_{P\in\mathcal{P}} E_P[X_i]$ and $\max_{P\in\mathcal{P}} E_P[X_i]$.



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0.5. Linear and Nonlinear Expectations

$\star$ Kolmogorov: Linear expectation:$P : \mathcal{F} \to [0,1]$, $P(A) = E[I_A]$

$$P(A+B) = P(A) + P(B), \;\; A \cap B = \emptyset \Leftrightarrow E[\xi + \eta] = E[\xi] + E[\eta]$$

Expectation is a linear functional of random variable.

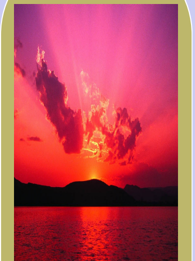
$\star$ Nonlinear probability(capacity): $V(\cdot) : \mathcal{F} \to [0,1]$ but

$$V(A+B) \neq V(A) + V(B), \text{even} A \cap B = \emptyset.$$

$\star$Nonlinear expectation: $\mathbb{E}(\xi)$ is nonlinear functional in the sense of

$$\mathbb{E}[\xi + \eta] \neq \mathbb{E}[\xi] + \mathbb{E}[\eta].$$

Capacity $V(A) = \mathbb{E}[I_A]$ is nonlinear.



Modes of nonlinear expectations and capacity

(1)Choquet expectations (Choquet 1953, physics)

$$C_V[X] := \int_0^{\infty} V(X \geq t)dt + \int_{-\infty}^{0} [V(X \geq t) - 1]dt.$$

(2)g-expectation (Peng 1997)

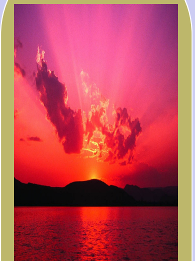
(3) Sub-linear expectation(Peng 2007).

(a)Monotonicity: $X \geq Y$ implies $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
(b)Constant preserving: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$.
(c)Sub-additivity: $\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
(d)Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.

(1) Distorted probability measure: $V(A) = g(P(A)), g : [0,1] \to [0,1]$.

(2) 2-alternating capacity: $V(A \cup B) \leq V(A) + V(B) - V(A \cap B)$

(3) $V(A) = \max_{P \in \mathcal{P}} P(A), \mathcal{P}$ set of Probability.



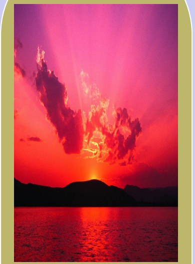
1. Independence w.r.t probability or capacity

⋆ Linear: A and B independent $P(AB) = P(A)P(B)$

$$\Leftrightarrow E[\phi(I_A + I_B)] = E[E[\phi(x + I_B)]|_{x=I_A}], \forall \phi(x)$$

⋆ Nonlinear: Epstein(2002), Marinacci(2005) $V(AB) = V(A)V(B)$

$$\Leftarrow \mathbb{E}[\phi(I_A + I_B)] = \mathbb{E}[\mathbb{E}[\phi(x + I_B)]|_{x=I_A}]$$



2. Definition of IID under expectation

DEFINITION 1 *(Peng 2007)*

Independence: *A random variable* $X \in \mathcal{H}$ *is said to be independent under* $\mathbb{E}$ *to Y, if for each* φ *such that* $\varphi(X, Y) \in \mathcal{H}$ *and* $\varphi(X, y) \in \mathcal{H}$ *for each* $y \in \mathbb{R}$

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\overline{\varphi}(Y)],$$

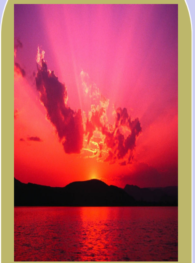
where $\overline{\varphi}(y) := \mathbb{E}[\varphi(X, y)]$.

Identical distribution: *Random variables* X *and* Y *are said to be identically distributed, if for each* φ *such that* $\varphi(X),\ \varphi(Y) \in \mathcal{H}$,

$$\mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)].$$

Mutual independence: X and Y are mutually independent

$$\mathbb{E}[\phi(X + Y)] = \mathbb{E}[\mathbb{E}[\phi(X + y)]|_{y=Y}]$$



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3. Definition: capacity and nonlinear expectation

(1) Probability space :$(\Omega, \mathcal{F}, P) \Rightarrow (\Omega, \mathcal{F}, \mathcal{P})$. Where $\mathcal{P} := \{P_\theta : \theta \in \Theta\}$.

(2) Capacity: $P \Rightarrow (v, V)$, where

$$v(A) = \inf_{Q \in \mathcal{P}} Q(A), \quad V(A) = \sup_{Q \in \mathcal{P}} Q(A).$$

(3)Property:

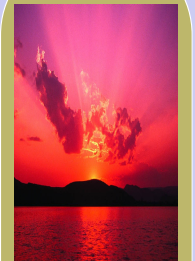
$$V(A) + V(A^c) \geq 1, \quad v(A) + v(A^c) \leq 1$$

but

$$V(A) + v(A^c) = 1.$$

(4) Nonlinear expectations: Lower-upper expectation $\mathcal{E}[\xi]$ and $\mathbb{E}[\xi]$

$$\mathcal{E}[\xi] = \inf_{Q \in \mathcal{P}} E_Q[\xi], \qquad \mathbb{E}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi]$$



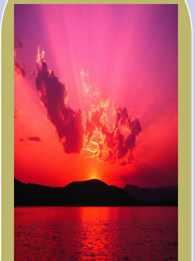
4. LLN for sub-linear expectations

⋆ Weak LLN:

THEOREM 2 *(Peng 2007,2008)* $\{X_i\}_{i=1}^{\infty}$ *IID random variables,* $\overline{\mu} := \mathbb{E}[X_1]$, $\underline{\mu} := \mathcal{E}[X_1]$. *Then for any continuous and linear growth function* ϕ,

$$\mathbb{E}\left[\phi\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right] \to \sup_{\underline{\mu}\le x\le\overline{\mu}} \phi(x), \quad \textit{as } n \to \infty.$$

⋆ Theorem (Peng, 2006,2007). CLT for IID



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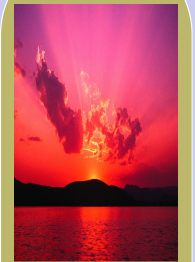
$\star\, V(AB) = V(A)V(B), v(AB) = v(A)v(B)$

$\star$ Theorem (Epstein, 02, Marinacci, 99, 05). ξ Bounded, Ω Polish,$C_v[X_i] = \underline{\mu}, C_V[X_i] = \overline{\mu}$. $\{X_i\}$ IID, then

$$v\left(\underline{\mu} \leq \liminf_{n\to\infty} S_n/n \leq \limsup_{n\to\infty} S_n/n \leq \overline{\mu}\}\right) = 1.$$

Where V is totally 2-alternating $V(A \bigcup B) \leq V(A) + V(B) - V(AB)$, here C_v and C_V is Choquet are integrals.

Note $C_v[X] \leq \mathcal{E}[X] \leq \mathbb{E}[X] \leq C_V[X], \forall X$.



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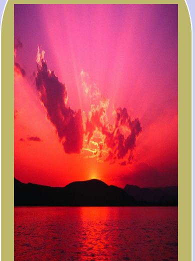
4.1. Limit theorem 1

Theorem: If $\{X_i\}$ is IID, then $\frac{S_n}{n}$ converges as $n \to \infty$ a.s. v if and only if

$$\mathcal{E}[X_1] = \mathbb{E}[X_1].$$

In this case,

$$\lim S_n/n = \mathcal{E}[X_1], \quad a.s. \quad v.$$



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5. Main results

THEOREM 3 $\{X_i\}_{i=1}^n$ *IID under nonlinear expectation* $\mathbb{E}$. *Set* $\overline{\mu} := \mathbb{E}[X_i]$, $\underline{\mu} := \mathcal{E}[X_i]$ *and* $S_n := \sum_{i=1}^n X_i$. *If* $\mathbb{E}[|X_i|^{1+\alpha}] < \infty$ *for* $\alpha > 0$. *Then*

(I)

$$v\left(\omega \in \Omega : \underline{\mu} \leq \liminf_{n\to\infty} S_n(\omega)/n \leq \limsup_{n\to\infty} S_n/n(\omega) \leq \overline{\mu}\}\right) = 1.$$

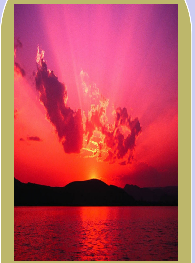
(II)

$$V\left(\omega \in \Omega : \limsup_{n\to\infty} S_n(\omega)/n = \overline{\mu}\right) = 1$$

$$V\left(\omega \in \Omega : \liminf_{n\to\infty} S_n(\omega)/n = \underline{\mu}\right) = 1.$$

(III) Suppose that $C(\{S_n(\omega)/n\})$ *is the cluster set of a sequence of* $\{S_n(\omega)/n\}$, *then*

$$V\left(\omega \in \Omega : C(\{S_n(\omega)/n\}) = [\underline{\mu}, \overline{\mu}]\right) = 1$$



6. Law of iterated logarithm for sub-linear expectations

THEOREM 4 $\{X_n\}$ *bounded IID.* $\mathbb{E}[X_1] = \mathcal{E}[X_1] = 0$, $\overline{\sigma}^2 := \mathbb{E}[X_1^2], \underline{\sigma}^2 := \mathcal{E}[X_1^2]$. *Let* $S_n := \sum_{i=1}^n X_i$, $a_n := \sqrt{2n \lg \lg n}$, *then*

(I)

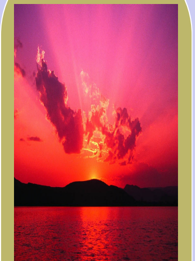
$$v\left(\underline{\sigma} \leq \limsup_n \frac{S_n}{a_n} \leq \overline{\sigma}\right) = 1;$$

(II)

$$v\left(-\overline{\sigma} \leq \liminf_n \frac{S_n}{a_n} \leq -\underline{\sigma}\right) = 1.$$

(III) Suppose that $C(\{x_n\})$ *is the cluster set of a sequence of* $\{x_n\}$ *in* R, *then*

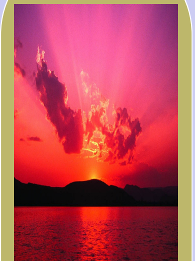
$$v\left(C(\{S_n/\sqrt{2n\mathrm{loglog}n}\}) \supset (-\underline{\sigma}, \underline{\sigma})\right) = 1.$$



7. Key of proof

THEOREM 5 *Suppose ξ is distributed to G normal $N(0;[\underline{\sigma}^2,\overline{\sigma}^2])$, where $0<\underline{\sigma}\le\overline{\sigma}<\infty$. Let ϕ be a bounded continuous function. Furthermore, if ϕ is a positively even function, then, for any $b\in R$,*

$$e^{-\frac{b^2}{2\underline{\sigma}^2}}\mathcal{E}[\phi(\xi)]\le\mathcal{E}[\phi(\xi-b)].$$



8. Application

Total 100 balls in box, Black + Red + Yellow $= 100$,

Black = Red, Yellow $\in [30, 40]$, then $P_Y \in [3/10, 4/10]$.

Take a ball from this box,

$X_i = 1$, if ball is black, $X_i = 0$, if ball is Yellow, $X_i = -1$ for red.

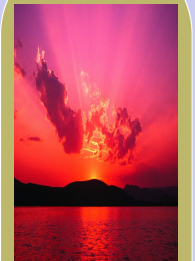
$S_n = \sum_{i=1}^{n} X_i$, is the excess frequency of black than Red

Then

(a) $\mathbb{E}[X_i] = \mathcal{E}[X_i] = 0$

(b)

$$\sqrt{6/10} \le \limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \lg \lg n}} \le \sqrt{7/10}.$$



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9. Nonlinear expectation in Finance

In incomplete markets, there exists a set $\mathcal{P}$ of probability measures, such that the super-sub-hedging price of option ξ at strike date T are given by $\underline{\mu} := \inf_{Q\in\mathcal{P}} E_Q[\xi], \overline{\mu} := \sup_{Q\in\mathcal{P}} E_Q[\xi]$.

then

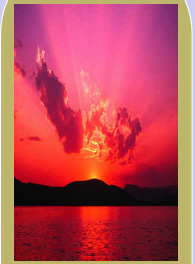
(1)

$$\underline{\mu} \le \liminf_{n\to\infty} S_n(\omega)/n \le \limsup_{n\to\infty} S_n/n(\omega) \le \overline{\mu}$$

(2)

$$\limsup_{n\to\infty} S_n(\omega)/n = \overline{\mu}, \quad V,$$

$$\liminf_{n\to\infty} S_n(\omega)/n = \underline{\mu}, \quad V$$



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