Endoscopic Relative Orbital Integrals on U_3

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Distinction of Automorphic Representations

Automorphic representations

Global setting: K is a number field, and G is a reductive group over K.

- Consider the vector space

$$V = L^2([G]) = L^2(G(K)\backslash G(\mathbb{A}_K)).$$

- Right regular representation: an action of G (thus a representation) $R : G(\mathbb{A}_K) \to \mathrm{GL}(V)$ by translation

$$R(g) \cdot \varphi(x) = \varphi(xg).$$

- Ultimate goal: to study the decomposition and constituents of (V, R). The isotypic components of V are called *automorphic representations*.

Distinction of automorphic representations

Let $H \subset G$ be a closed subgroup.

Definition

1. *For any automorphic representation* π *and* $\varphi \in V_\pi$*, define the* ***period integral*** *of* φ *with respect to* H *to be*

$$P_H(\varphi) = \int_{[H]} \varphi(h)\mathrm{d}h.$$

2. *An automorphic representation* π *is* H***-distinguished*** *if* $P_H(\varphi) \neq 0$ *for some* $\varphi \in V_\pi$*.*

Examples of distinguished representations

1. H: a maximal unipotent. Then cuspidal representations $\Leftrightarrow$ not distinguished by H.
2. L/K is quadratic. $G = \mathrm{Res}_{L/K}\mathrm{GL}_n$ and $H = \mathrm{U}_n$. Then H-distinguished $\Leftrightarrow$ a base change lifting from GL_n (**Flicker**, **Mok** and **Zinoviev**).
3. $G = \mathrm{GL}_n$ and $H = \mathrm{GO}_n$. Then H-distinguished $\Leftrightarrow$ a metaplectic lifting from $\widetilde{\mathrm{GL}_n}$ (conjecture of **Jacquet**, verified in $n = 3$ by **Mao**).
4. Distinction is also related to central L-values.

Motivation behind distinction problem

Distinction problem is fruitful from several perspectives:

1. The subject of the relative Langlands program introduced by Sakellaridis and Venkatesh.
2. To study algebraic cycles on Shimura varieties.
3. Related to the Ichino-Ikeda conjecture and its generalizations.

The Comparison of Relative Trace Formulae

A strategy for distinction

- Philosophy (**Jacquet**): to study distinction problems through the *comparison of relative trace formulae*.
- The proof of the unitary case of Gan-Gross-Prasad conjecture.
- An approach by **Getz-Wambach** that considers a triple of involutions.
  Distinction problems on classical groups $\longrightarrow$ Distinction problems on general linear groups.

The automorphic kernel

- For $f \in C_c^\infty(G(\mathbb{A}_K))$, we can define an operator $R(f)$ via

$$R(f)\varphi(x) = \int\limits_{G(\mathbb{A}_K)} f(y)\varphi(xy)\mathrm{d}y.$$

- The operator $R(f)$ has a kernel

$$K_f(x,y) = \sum_{\gamma \in G(K)} f(x^{-1}\gamma y)$$

referred to as the *automorphic kernel* associated to f.

- With some additional conditions, the automorphic kernel allows a *spectral expansion*

$$K_f(x,y) = \sum_{\pi \in \hat{G}} m(\pi) \sum_{\phi \in \mathcal{B}_\pi} \pi(f)\phi(x)\overline{\phi(y)}.$$

Relative trace: the spectral side

Let G_1, G_2 be two algebraic subgroups of the reductive group G.

- We consider $\int_{[G_1 \times G_2]} K_f(x, y) \mathrm{d}x \mathrm{d}y$.
- If we uses the spectral expansion for $K_f(x, y)$, this integral is equal to

$$\sum_{\pi \in \hat{G}} m(\pi) \sum_{\phi} \int_{[G_1]} \int_{[G_2]} \pi(f)\phi(x)\overline{\phi(y)} \mathrm{d}x \mathrm{d}y$$

$$= \sum_{\pi \in \hat{G}} m(\pi) J_\pi(f).$$

Relative trace: the geometric side

- On the other hand, the *geometric expansion* considers the same integral, but decomposed by equivalence classes.
- It becomes a sum over classes $\gamma \in G_1(K)\backslash G(K)/G_2(K)$:

$$\sum_{\gamma} a(\gamma) O_\gamma(f).$$

Here

$$O_\gamma(f) = \int_{I_\gamma(\mathbb{A}_K)\backslash G_1 \times G_2(\mathbb{A}_K)} f(x^{-1}\gamma y)\mathrm{d}x\mathrm{d}y$$

is the *relative orbital integral*.

The relative trace formula

- The *relative trace formula* thus asserts (roughly)

$$\sum_{\pi} m(\pi) J_\pi(f) = \sum_{\gamma} a(\gamma) O_\gamma(f),$$

- Here each $J_\pi(f)$ are related to the period integrals of functions in V_π with respect to G_1 and G_2 because

$$J_\pi(f) = \sum_{\phi \in \mathcal{B}_\pi} P_{G_1}(f \cdot \phi) \overline{P_{G_2}(\phi)}.$$

- Assume $f = \prod_v f_v$, so that the relative orbital integrals $O_\gamma(f)$ is a product of its local factors.
- Strategy: using the (local) comparison on the geometric side to study the spectral data.

The Getz-Wambach comparison

- Setting: let L/K be a quadratic extension of number fields and $H = \mathrm{Res}_{L/K}\mathrm{GL}_n$.
- Given a pair of commuting involutions (automorphism of order 2) on H: θ and σ. Let $\tau = \sigma \circ \theta$.
- Twisted relative trace formula of $H \longleftrightarrow$ Relative trace formula of $G = H^\tau$.

A general principle

- With the previous setting, they suggested that (roughly speaking) one should have

Ansatz (Getz-Wambach)

An automorphic representation π of $H(\mathbb{A}_K)$ is distinguished by both H^σ and H^θ $\Leftrightarrow$ π is a lifting of an G^σ-distinguished automorphic representation on $G(\mathbb{A}_K)$.

They proposed a relative trace formula method of proof.

- Relates directly to the relative Langlands program because symmetric subgroups are always spherical.

Studied examples

- The biquadratic case: a theorem of **Getz-Wambach** says that (with some additional condition) the ansatz holds for the case of $\mathrm{U}_n \subset \mathrm{Res}_{L/K}\mathrm{U}_n$, where L is a quadratic extension over K.
- The unitary Friedberg-Jacquet case: for $\mathrm{U}_n \times \mathrm{U}_n \subset \mathrm{U}_{2n}$.
  - ($\Rightarrow$) The work of **S. Leslie** and the work of **J. Xiao** and **W. Zhang**.
  - ($\Leftarrow$) The work of **Pollack-Wan-Zydor**.

Case of interest and its conjecture

- Let L/K be a quadratic extension. $H = \mathrm{Res}_{L/K}\mathrm{GL}_n$, σ be a quasi-split orthogonal involution, and θ be the nontrivial Galois conjugate.
- Thus $G = \mathrm{U}_n$ and $G^\sigma = \mathrm{O}_n$ are quasi-split reductive groups.

Conjecture (L.)

A cuspidal automorphic representation π of $\mathrm{U}_n(\mathbb{A}_K)$ is distinguished by O_n $\Leftrightarrow$ its base change lifting to $\mathrm{Res}_{L/K}\mathrm{GL}_n(\mathbb{A}_K)$ is a metaplectic lifting from $\mathrm{Res}_{L/K}\widetilde{\mathrm{GL}_n}(\mathbb{A}_K)$.

The associated symmetric variety

- We consider the integrals on $H^\sigma \backslash H / H^\theta$ and the integrals on $G^\sigma \backslash G / G^\sigma$. Furthermore, we will be focusing on the latter for the rest of the talk.
- The symmetrization map

$$\begin{aligned} G &\to G \\ g &\mapsto gg^{-\sigma} \end{aligned}$$

  has kernel G^σ.
- Instead of considering G/G^σ, we consider the schematic image of this map, denoted as $\mathcal{S}$.
- $\mathcal{S}$ is a spherical variety under the adjoint action of G.
- Studying setting: fix $n = 3$ and consider the adjoint action of $G_1 := \mathrm{SO}_3$ on the symmetric space $\mathcal{S}$.

Pre-stabilization and Endoscopy

The pre-stabilization of the relative trace formulae

The adelic ring: $\mathbb{A}_K = \prod_v^{\text{res}} K_v$. Fix a nonarchimedean local field $K_v = F$ and denote its valuation ring by $\mathcal{O}_F$.

- On the geometric side of the relative trace formula: relative orbital integrals with local factors

$$O_\gamma(f) = \int_{G_{1\gamma}(F)\backslash G_1(F)} f(g^{-1}\cdot\gamma)\mathrm{d}g.$$

- Over the algebraic closure, there is a *norm map* to match orbits.
- Difficulty: the F-points of $G_{1\gamma}\backslash G_1$ can be different to $G_{1\gamma}(F)\backslash G_1(F)$.
- The solution to this issue is called the *pre-stabilization*.

Stable orbits versus rational orbits

- Stable orbits: F-points of $G_{1\gamma}\backslash G_1 \longleftrightarrow G_1(\overline{F})$-classes in $\mathcal{S}(F)$.
- Rational orbits: $G_{1\gamma}(F)\backslash G_1(F) \longleftrightarrow G_1(F)$-classes.
- The set of rational orbits inside a stable orbit is parametrized by a group $\mathfrak{D}(F, G_{1\gamma}, G_1) := \ker[H^1(F, G_{1\gamma)} \to H^1(F, G_1)]$.
- Harmonic analysis: to stabilize, one should consider the local κ-orbital integrals.

Relative κ-orbital integrals

- For each $\kappa \in \mathfrak{D}(F, G_{1\gamma}, G_1)^{\mathrm{D}}$, define the κ-orbital integral
$$SO_\gamma^\kappa(f) = \sum_{\gamma' \sim_{\mathrm{st}} \gamma} \kappa(\gamma') O_\gamma(f).$$
- $SO_\gamma(f) := SO_\gamma^1(f)$ is called the *stable relative orbital integral*.
- We say that $SO_\gamma^\kappa(f)$ is an *endoscopic relative orbital integral* if κ is nontrivial.
- Expectation: proceed inductively by relating the endoscopic relative orbital integrals to stable relative orbital integrals on other (simpler) spaces.

Endoscopic Relative Orbital Integrals

Unitary relative endoscopy

Local setting: let $E = F[\xi]$ be an unramified quadratic extension of local fields. Denote the uniformizer by ϖ.

- Recall: $G = \mathrm{U}_3$ and $G^\sigma = \mathrm{O}_3$ are quasi-split.
  $\mathcal{S}$ is the space of (twisted) symmetric unitary matrices.
  $G_1 := \mathrm{SO}_3$.
- The generic stabilizer $G_{1\gamma}$ is *disconnected* and *finite*.
- For any regular semisimple point $\gamma \in \mathcal{S}(F)$, we want to compute $SO^\kappa_\gamma(f)$ for $f = \mathbb{1}_{\mathcal{S}(\mathcal{O}_F)}$ for κ nontrivial.
- In particular, we consider only those γ with a nontrivial $\mathfrak{D}(F, G_{1\gamma}, G_1)$.

Classification of stable orbits

- It turns out that $\mathfrak{D}(\gamma, G_{1\gamma}, G_1)$ can be computed by the isomorphism classes of G_γ, which occurs in **J. Rogawski**'s work.
- Let

$$T_\nu(R) = \left\{ \begin{pmatrix} x & & y \\ & z & \\ \nu y & & x \end{pmatrix} \in \mathrm{U}_3(R) \mid x, y, z \in E \otimes_F R \right\}.$$

Lemma (L.)

Any regular semisimple stable orbit with a nontrivial $\mathfrak{D}(F, G_{1\gamma}, G_1)$ contains an element in (some unique) $T_\nu(F)$ with $\nu \in \{1, \xi^2, \varpi, \xi^2\varpi\}$.

Types of the endoscopic stable orbits

- The stable orbits of interest are represented by elements in $T_\nu(F)$ for some $\nu \in \{1, \xi^2, \varpi, \xi^2\varpi\}$.
- Types of tori: we say that T_ν is of

$$\begin{cases} \text{Type I} & \text{if } \nu = \xi^2, \\ \text{Type II} & \text{if } \nu = 1, \\ \text{Type III} & \text{if } \nu = \varpi \text{ or } \xi^2\varpi. \end{cases}$$

- Later, we will compute the formula on type I tori as an explicit example.

Related cohomological data

- Recall: $\mathfrak{D}(F, G_{1\gamma}, G_1) = \ker[H^1(F, G_{1\gamma}) \to H^1(F, G_1)]$ parametrizes rational orbits inside the stable orbits of γ. In general, those are *Galois cohomology pointed sets*.

Lemma (L.)

In our case, there exist a natural group structure on $\mathfrak{D}(F, G_{1\gamma}, G_1)$ *so that*

$$\mathfrak{D}(F, G_{1\gamma}, G_1) \cong \begin{cases} F^\times \backslash \mathrm{N}(E^\times) & \text{for Type I tori,} \\ F^\times \backslash (F^\times)^2 & \text{for Type II and III tori.} \end{cases}$$

- In particular, $|\mathfrak{D}(F, G_{1\gamma}, G_1)| = 2$ for γ in a Type I torus.

Iwasawa decomposition

- Recall that we are considering

$$O_\gamma(f) = \int_{G_{1\gamma}(F)\backslash G_1(F)} \mathbb{1}_{\mathcal{S}(\mathcal{O}_F)}(g^{-1}\cdot\gamma)\mathrm{d}g.$$

- Iwasawa decomposition: $G(F) = N(F)A(F)G(\mathcal{O}_F)$. Since f is $G(\mathcal{O}_F)$-invariant we have (with the suitable choice of measure)

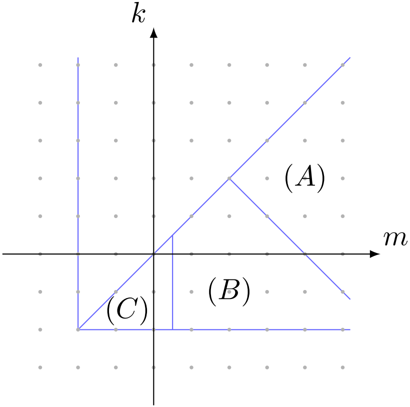
$$O_\gamma(f) = \int_{F\times F^\times} f\left(\begin{pmatrix} t & u & -t^{-1}u^2/2 \\ & 1 & -t^{-1}u \\ & & t^{-1} \end{pmatrix}^{-1}\cdot\gamma\right)\frac{\mathrm{d}u\mathrm{d}^\times t}{|t|}.$$

Relative orbital integral

- Rational orbits in the stable orbit of γ are given by $\gamma_\mu = \begin{pmatrix} & x & & \mu y \\ & & z & \\ \mu^{-1}\nu y & & & x \end{pmatrix}$.
- The entries of $\begin{pmatrix} t & u & -t^{-1}u^2/2 \\ & 1 & -t^{-1}u \\ & & t^{-1} \end{pmatrix}^{-1} \cdot \gamma_\mu$ are (excluding the repeated ones):
  1. $x - \frac{1}{2}u^2\mu^{-1}\nu y$,
  2. $t^{-1}(ux - uz - \frac{1}{2}u^3\mu^{-1}\nu y)$,
  3. $t^{-2}(\mu y - u^2 x + u^2 z + \frac{1}{4}u^4\mu^{-1}\nu y)$,
  4. $tu\mu^{-1}\nu y$,
  5. $u^2\mu^{-1}\nu y + z$,
  6. $t^2\mu^{-1}\nu y$.
- Problem: computing the measure of those u and t that makes all these entries integral.

A few technical remarks

- Notice that by Iwasawa decomposition, the orbital integral can be reduced to a double integral taken over $F \times F^{\times}$.
- We will write $v(t) = m$ and $v(u) = k$, then separate the orbital integral accordingly:



Some invariants

- Goal: express the relative orbital integrals in terms of invariants.
- Let λ_i denote the eigenvalues of γ (we fix an ordering so that $\lambda_2 = z$).
- Invariants associated to the stable orbit of γ:

$$M_{ij} := v(\lambda_i - \lambda_j),$$
$$N_{ij} := v(\lambda_i + \lambda_j).$$

Integration

The nature of computation varies on different parts of the orbital integral.

- On (A), it involves solving a quadratic equation

$$u^4 \equiv 4\mu^2\xi^{-2} \pmod{\varpi^{m+k-M_{13}+v(\mu)}}.$$

- On (B), it also depends on solving a quadratic equation, along with a combinatorial datum $2k + 2M_{12} \geq 2m > 2k + M_{12}$.
- On (C) all the conditions are combinatorial: $2M_{12} - M_{13} + v(\mu) \geq 2m > -M_{13} + v(\mu)$ and $4m > 4k \geq 2m - M_{13} + v(\mu)$.

The formula for relative orbital integrals

- The expressions for $O_{\gamma_\mu}(f)$ depends on M_{ij}.
- I have computed the formula for $O_{\gamma_\mu}(f)$ for any γ_μ.
- For instance, when $M_{13} > M_{12} > 0$, $O_{\gamma_\mu}(\mathbb{1}_{\mathcal{S}(\mathcal{O}_F)})$ is equal to

$$\frac{1}{2}\Big((M_{13} - M_{12} + \delta(M_{12}, 1))\, q^{\lfloor M_{12}/2 \rfloor} + 2\,(1 + \delta(M_{12} - M_{13}, \mu))\, \frac{q^{\lceil M_{12}/2 \rceil} - 1}{q-1} + \delta(M_{13}, \mu) - 1 \Big).$$

Here δ is a function that detects parity.

The formula

For every γ in type I, II and III tori I have computed the formula for endoscopic relative orbital integrals. In particular,

Theorem (L.)

Let γ be in a type I torus. The endoscopic orbital integral $SO_\gamma^\kappa(\mathbb{1}_{\mathcal{S}(\mathcal{O}_F)})$ is computed as in the following table:

	$\kappa \neq 1$
$M_{13} = 0$	$\frac{1}{2}$
$M_{13} > M_{12} = 0$	$\frac{1}{2}(-1)^{M_{13}}$
$M_{13} > M_{12} > 0$	$(-1)^{M_{12}-M_{13}}\frac{q^{\lceil M_{12}/2\rceil}-1}{q-1} + \frac{1}{2}(-1)^{M_{13}}$
$M_{12} = M_{13} = M_{23} > 0$	$\frac{1}{2}(1+\left(\frac{z_y^2-\xi^2}{F}\right))\frac{q^{\lfloor M_{12}/2\rfloor}-1}{q-1} + \frac{1}{2}(-1)^{M_{12}}$

Thank you for your attention.