A BLOW UP FORMULA FOR GYSIN PULL-BACK

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Abstract. In this note, we prove a blow up formula for Gysin pull-back of cycles by the zero section of a cotangent bundle (cf. Lemma 3.3). A special case of this formula is used in the proof of the twist formula for ε-factors [6].

1. Preliminaries on C-transversal condition

Definition 1.1. *Let X, Y and W be smooth schemes over a field k. We denote by $T^*_X X \subseteq T^*X$ the zero section of the cotangent bundle T^*X of X. Let C be a conical closed subset of T^*X, i.e., a closed subset which is stable under the action of the multiplicative group $\mathbb{G}_m$.*

(1) ([2, 1.2]) *Let $h\colon W \to X$ be a morphism over k. We say that h is C-transversal at $w \in W$ if the fiber $\big((C \times_X W) \cap dh^{-1}(T^*_W W)\big) \times_W w$ is contained in the zero-section $T^*_X X \times_X W \subseteq T^*X \times_X W$, where $dh\colon T^*X \times_X W \to T^*W$ is the canonical map. We say that h is C-transversal if h is C-transversal at any point of W.*

If h is C-transversal, we define $h^\circ C$ to be the image of $C \times_X W$ under the map $dh\colon T^*X \times_X W \to T^*W$. By [5, Lemma 3.1], *$h^\circ C$ is a closed conical subset of T^*W.*

(2) ([5, Definition 7.1]) *Assume that X and C are purely of dimension d and that W is purely of dimension m. We say that a C-transversal map $h\colon W \to X$ is properly C-transversal if every irreducible component of $C \times_X W$ is of dimension m.*

(3) ([2, 1.2] *and* [5, Definition 5.3]) *We say that a morphism $f\colon X \to Y$ over k is C-transversal at $x \in X$ if the inverse image $df^{-1}(C) \times_X x$ is contained in the zero-section $T^*_Y Y \times_Y X \subseteq T^*Y \times_Y X$, where $df\colon T^*Y \times_Y X \to T^*X$ is the canonical map. We say that f is C-transversal if f is C-transversal at any point of X.*

1.2. Let X be a smooth scheme purely of dimension d over a field k. Let W be a smooth scheme purely of dimension m over k. Assume that $C \subseteq T^*X$ is a conical closed subset purely of dimension d. Let Z be a d-cycle supported on C and $h\colon W \to X$ a properly C-transversal morphism. Let $\mathrm{pr}_h\colon T^*X \times_X W \to T^*X$ be the first projection map. Since pr_h is a morphism between smooth schemes, the refined Gysin pull-back $\mathrm{pr}_h^! Z$ is well-defined in the sense of intersection theory [3, 6.6]. We define $h^*Z \in CH_m(h^\circ C)$ [5, Definition 7.1.2] to be

$$h^*Z := dh_*(\mathrm{pr}_h^! Z). \tag{1.2.1}$$

Notice that the push-forward is well-defined since $dh\colon T^*X \times_X W \to T^*W$ is finite on $C \times_X W$ by [2, Lemma 1.2 (ii)]. Since h is properly C-transversal, every irreducible component of $h^\circ C$ is of dimension m. Thus $CH_m(h^\circ C) = Z_m(h^\circ C)$. Hence we may regard h^*Z as a m-cycle on T^*W, which is supported on $h^\circ C$.

We prove the following commutative property for successively pull-backs.

Lemma 1.3. *Let X be a smooth scheme purely of dimension d over a field k. Consider the following commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{j} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array} \tag{1.3.1}$$

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between equidimensional smooth schemes over k. Let $C \subseteq T^*X$ be a conical closed subset purely of dimension d. Assume that i and f are C-transversal, and g is $i^\circ C$-transversal. Let Z be a d-cycle supported on C. Then we have

(1) j is $f^\circ C$-transversal.

(2) $g^\circ i^\circ C = j^\circ f^\circ C \subseteq T^*U$ and an equality for cycle class $g^*i^*Z = j^*f^*Z$.

Proof. (1) This follows from [5, Lemma 3.4.3].

(2) We have a commutative diagram

$$\begin{array}{ccccc} T^*X & \xleftarrow{\mathrm{pr}_f} & T^*X \times_X W & \xrightarrow{df} & T^*W \\ \uparrow{\scriptstyle \mathrm{pr}_i} & & \uparrow{\scriptstyle v} & \square & \uparrow{\scriptstyle \mathrm{pr}_j} \\ T^*X \times_X Y & \xleftarrow{u} & T^*X \times_X U & \xrightarrow{w} & T^*W \times_W U \\ \downarrow{\scriptstyle di} & \square & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle dj} \\ T^*Y & \xleftarrow{\mathrm{pr}_g} & T^*Y \times_Y U & \xrightarrow{dg} & T^*U \end{array} \tag{1.3.2}$$

where the morphisms $\mathrm{pr}_f, \mathrm{pr}_g, \mathrm{pr}_i$ and pr_j are the first projections, df, dg, di, dj are morphisms induced from f, g, i, j respectively, and $v = \mathrm{id} \times j, u = \mathrm{id} \times g, r = di \times \mathrm{id}$. In the diagram (1.3.2), there are two Cartesian squares which are indicated by the symbols "$\square$". Then we have

$$\begin{aligned} g^\circ i^\circ C &= dg(\mathrm{pr}_g^{-1}(di(\mathrm{pr}_i^{-1}C))) = dg(r(u^{-1}(\mathrm{pr}_i^{-1}C))) \\ &= dg(r(v^{-1}(\mathrm{pr}_f^{-1}C))) = dj(\omega(v^{-1}(\mathrm{pr}_f^{-1}C))) \\ &= dj(\mathrm{pr}_j^{-1}(df(\mathrm{pr}_f^{-1}C))) = j^\circ f^\circ C. \end{aligned} \tag{1.3.3}$$

$$\begin{aligned} g^*i^*Z &= dg_*(\mathrm{pr}_g^!(di_*(\mathrm{pr}_i^!Z))) = dg_*(r_*(u^!(\mathrm{pr}_i^!Z))) \\ &= dg_*(r_*(v^!(\mathrm{pr}_f^!Z))) = dj_*(\omega_*(v^!(\mathrm{pr}_f^!Z))) \\ &= dj_*(\mathrm{pr}_j^!(df_*(\mathrm{pr}_f^!Z))) = j^*f^*Z \end{aligned} \tag{1.3.4}$$

where in (1.3.4) we used the push-forward formula [3, Theorem 6.2 (a)] and the fact that di (respectively df) is finite on $\mathrm{pr}_i^!Z$ (respectively $\mathrm{pr}_f^!Z$). This finishes the proof. $\square$

2. Localized Chern classes

2.1. Let X be a scheme of finite type over a field k, Z a closed subscheme of X and $U = X \backslash Z$. Let $\mathcal{K} = (\mathcal{K}_q, d_q)_q$ be a bounded complex of locally free $\mathcal{O}_X$ -modules of finite ranks such that $\mathcal{K}_q = 0$ for $q < 0$. Assume that the restriction $\mathcal{K}|_U$ is acyclic except at degree 0 and the cohomology sheaf $\mathcal{H}_0(\mathcal{K})|_U$ is locally free of rank $n-1$. Then for $i \geqslant n$, we have the so-called localized Chern class ${c_i}_Z^X(\mathcal{K}) \in CH^i(Z \to X)$ (cf. [1, Section 3], [3, Chapter 18] and [4, 2.3]). Consider the following ring (cf. [3, Chapter 17])

$$CH^*(Z \to X)^{(n)} = \prod_{i<n} CH^i(X \to X) \times \prod_{i \geqslant n} CH^i(Z \to X). \tag{2.1.1}$$

We regard the total localized Chern class $c_Z^X(\mathcal{K}) = ((c_i(\mathcal{K}))_{i<n}, ({c_i}_Z^X(\mathcal{K}))_{i \geqslant n})$ as an invertible element of $CH^*(Z \to X)^{(n)}$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that the restriction $\mathcal{F}|_U$ is locally free of rank n. If $\mathcal{F}$ has a finite resolution $\mathcal{E}_\bullet \to \mathcal{F}$ by locally free $\mathcal{O}_X$-modules $\mathcal{E}_q$ of finite ranks, the localized Chern class ${c_i}_Z^X(\mathcal{F})$ for $i > n$ is defined as ${c_i}_Z^X(\mathcal{E}_\bullet)$. It is independent of the choice of a resolution.

2.2. The following Lemma 2.3 and Lemma 2.4 are slight generalizations of [4, Lemma 2.3.2] and [4, Lemma 2.3.4] respectively. We use the same arguments.

Lemma 2.3 ([4])**.** *Let X be a scheme of finite type over a field k. Let D be a Cartier divisor of X and $i \colon D \to X$ be the immersion. Let $\mathcal{E}$ be a locally free $\mathcal{O}_D$-module of rank n. Assume there exist a locally free $\mathcal{O}_X$-module $\widetilde{\mathcal{E}}$ of finite rank and a surjection $\widetilde{\mathcal{E}} \to i_*\mathcal{E}$ so that the localized Chern class $c_D^X(i_*\mathcal{E}(D)) \in CH^*(D \to X)^{(1)}$ is defined. We put $CH_*(X) = \oplus_i CH_i(X)$, $CH_*(D) = \oplus_i CH_i(D)$ and put $a_j(\mathcal{E}) = \sum_{k=j}^n \binom{k}{j} c_{n-k}(\mathcal{E}) \in CH^*(D \to D)$.*

(1) ([4, Lemma 2.3.2]) *For any invertible $\mathcal{O}_D$-module $\mathcal{L}$, we have*

$$\sum_{k=0}^{n} c_k(\mathcal{E}\otimes\mathcal{L}) = \sum_{j=0}^{n} a_j(\mathcal{E})c_1(\mathcal{L})^j. \tag{2.3.1}$$

(2) ([4, Lemma 2.3.2 and Corollary 2.3.2]) *For any $\alpha\in CH_*(X)$, we have equalities in $CH_*(D)$:*

$$(c_D^X(i_*\mathcal{E}(D)) - 1)\cap\alpha = c(\mathcal{E})^{-1}\sum_{j=1}^{n} a_j(\mathcal{E})D^{j-1}\cap i^!\alpha, \tag{2.3.2}$$

$$(c_D^X(i_*\mathcal{O}_D)^{-1} - 1)\cap\alpha = -i^!\alpha. \tag{2.3.3}$$

where $i^!\colon CH_*(X)\to CH_*(D)$ denotes the Gysin map.

Proof. (1) See [4, Lemma 2.3.2].

(2) We use the same argument with [4, Lemma 2.3.2]. By deformation to the normal bundle, we may assume $X = \mathbb{P}^1_D$ is a $\mathbb{P}^1$-bundle over D and the immersion $i\colon D\to X$ is a section. Let $p\colon X\to D$ be the projection. Then $\mathcal{E} = i^*\mathcal{E}_X$ with $\mathcal{E}_X := p^*\mathcal{E}$. Since the map $i_* : CH_*(D)\to CH_*(X)$ is injective, it is reduced to the equalities for the usual Chern classes $c(i_*\mathcal{E}(D))$ and $c(i_*\mathcal{O}_D)$ by [4, Proposition 2.3.1.1]. By the exact sequence

$$0\to\mathcal{O}_X(-D)\to\mathcal{O}_X\to i_*\mathcal{O}_D\to 0, \tag{2.3.4}$$

we get $c(i_*\mathcal{O}_D)^{-1} = c(\mathcal{O}_X(-D))$. Thus $(c(i_*\mathcal{O}_D)^{-1} - 1)\cap\alpha = -c_1(\mathcal{O}_X(D))\cap\alpha = -i^!\alpha$. This proves the equality (2.3.3). Now we prove (2.3.2). By the locally free resolution

$$0\to\mathcal{E}_X\to\mathcal{E}_X(D)\to i_*\mathcal{E}(D)\to 0, \tag{2.3.5}$$

we have

$$\begin{aligned} c(i_*\mathcal{E}(D)) - 1 &= c(\mathcal{E}_X)^{-1}(c(\mathcal{E}_X(D)) - c(\mathcal{E}_X)) \\ &\overset{(2.3.1)}{=} c(\mathcal{E})^{-1}\Big(\sum_{j=0}^{n} a_j(\mathcal{E})D^j - a_0(\mathcal{E})\Big) = c(\mathcal{E})^{-1}\sum_{j=1}^{n} a_j(\mathcal{E})D^j. \end{aligned} \tag{2.3.6}$$

Thus by the definition of Gysin pull-back along a divisor [3, 2.6], we have

$$(c(i_*\mathcal{E}(D)) - 1)\cap\alpha = c(\mathcal{E})^{-1}\left(\sum_{j=1}^{n} a_j(\mathcal{E})D^j\right)\cap\alpha = c(\mathcal{E})^{-1}\left(\sum_{j=1}^{n} a_j(\mathcal{E})D^{j-1}\right)\cap i^!\alpha \tag{2.3.7}$$

□

Lemma 2.4 ([4, Lemma 2.3.4])**.** *Let X and C be regular schemes of finite type over a field k. Let $i\colon C\to X$ be a closed immersion of codimension c with conormal sheaf $N_{C/X}$. Let $\pi\colon X'\to X$ be the blow up of X along C, $\pi_E\colon E = C\times_X X'\to C$ be the induced map and $i'\colon E\to X'$ be the closed immersion. We put*

$$\Phi(X,C) = \sum_{j=1}^{c} a_j(\pi_E^* N_{C/X})E^{j-1} - \sum_{j=0}^{c} a_j(\pi_E^* N_{C/X})E^j. \tag{2.4.1}$$

For any $\alpha\in CH_*(X')$, we have an equality in $CH_*(C)$:

$$\pi_{E*}((c_E^{X'}(\Omega^1_{X'/X}) - 1)\cap\alpha) = c(N_{C/X})^{-1}\cap\pi_{E*}(\Phi(X,C)\cap i'^!\alpha), \tag{2.4.2}$$

If moreover $i'^!\alpha = \pi_E^*\beta$ for some $\beta\in CH_*(C)$, then we have

$$\pi_{E*}(\Phi(X,C)\cap i'^!\alpha) = (-1)^c\cdot(c-1)\cdot\beta, \tag{2.4.3}$$

$$\pi_{E*}((c_E^{X'}(\Omega^1_{X'/X}) - 1)\cap\alpha) = (-1)^c\cdot(c-1)\cdot c(N_{C/X})^{-1}\cap\beta. \tag{2.4.4}$$

Proof. Note that the canonical map $\Omega^1_{X'/X}\to i'_*\Omega^1_{E/C}$ is an isomorphism. Since $E = \mathbb{P}((N_{C/X})^\vee)$ is a $\mathbb{P}^{c-1}$-bundle over C, we have an exact sequence $0\to\Omega^1_{E/C}\to\pi_E^* N_{C/X}(-1)\to\mathcal{O}_E\to 0$. Hence, we have $c_E^{X'}(\Omega^1_{X'/X}) = c_E^{X'}(i'_*\pi_E^* N_{C/X}(-1))c_E^{X'}(i'_*\mathcal{O}_E)^{-1}$. By the exact sequence $0\to\mathcal{O}_{X'}(-E)\to\mathcal{O}_{X'}\to i'_*\mathcal{O}_E\to 0$, we get

$$0\to\pi_E^* N_{C/X}\to\pi_E^* N_{C/X}(E)\to\pi_E^* N_{C/X}(E)\to 0. \tag{2.4.5}$$

By Lemma 2.3, we have

$$
\begin{aligned}
(2.4.6)\qquad & (c_E^{X'}(i'_*\pi_E^*N_{C/X}(-1))c_E^{X'}(i'_*\mathcal{O}_E)^{-1}-1)\cap\alpha \\
&= (c_E^{X'}(i'_*\pi_E^*N_{C/X}(-1))-1)\cap\alpha + c_E^{X'}(i'_*\pi_E^*N_{C/X}(-1))(c_E^{X'}(i'_*\mathcal{O}_E)^{-1}-1)\cap\alpha \\
&\overset{(2.3.3)}{=} (c_E^{X'}(i'_*\pi_E^*N_{C/X}(-1))-1)\cap\alpha - c(\pi_E^*N_{C/X}(-1))\cap i'^!\alpha \\
&\overset{(2.4.5)}{=} (c_E^{X'}(i'_*\pi_E^*N_{C/X}(E))-1)\cap\alpha - c_E(\pi_E^*N_{C/X})^{-1}c_E(\pi_E^*N_{C/X}(E))\cap i'^!\alpha \\
&\overset{(2.3.2)}{=} c_E(\pi_E^*N_{C/X})^{-1}\left(\sum_{j=1}^{c}a_j(\pi_E^*N_{C/X})E^{j-1}\cap i'^!\alpha - \sum_{j=0}^{c}a_j(\pi_E^*N_{C/X})E^{j}\cap i'^!\alpha\right) \\
&= c(N_{C/X})^{-1}\cap\pi_{E*}(\Phi(X,C)\cap i'^!\alpha).
\end{aligned}
$$

By [3, Remark 3.2.4, p.55], we have $E^c = -\sum_{j=1}^{c} c_j(\pi_E^*N_{C/X})E^{c-j}$. Assume $i'^!\alpha = \pi_E^*\beta$ for some $\beta\in CH_*(C)$. By [3, Proposition 3.1 (a)], we have $\pi_{E*}(E^j\cap\pi_E^*\beta)=0$ for $j<c-1$ and $\pi_{E*}(E^{c-1}\cap\pi_E^*\beta) = (-1)^{c-1}\beta$. Substituting these identities, we have

$$
\begin{aligned}
(2.4.7)\qquad & \pi_{E*}(\Phi(X,C)\cap i'^!\alpha) \\
&=(-1)^{c-1}\cdot(a_c(N_{C/X})-a_{c-1}(N_{C/X})+a_c(N_{C/X})c_1(N_{C/X}))\cap\beta
\end{aligned}
$$

Since $a_c(N_{C/X})=1$, $a_{c-1}(N_{C/X}) = c+c_1(N_{C/X})$ then $\pi_{E*}(\Phi(X,C)\cap i'^!\alpha) = (-1)^c\cdot(c-1)\cdot\beta$ and $\pi_{E*}((c_E^{X'}(\Omega^1_{X'/X})-1)\cap\alpha) = (-1)^c\cdot(c-1)\cdot c(N_{C/X})^{-1}\cap\beta$. □

3. Blow up formula for Gysin pull-back

3.1. Let X be a smooth scheme purely of dimension d over a field k. We denote by $0_X\colon X\to T^*X$ the zero section of the cotangent bundle T^*X. We denote by $0_X^!\in CH^d(X\to T^*X)$ the (refined) Gysin map [3, 6.2], where $CH^d(X\to T^*X)$ is the bivariant Chow group [3, Definition 17.1]).

3.2. We recall a method for calculating the Gysin map $0_X^!$ by using Chern classes. Let X be a regular scheme separated of finite type over a field. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-modules of rank d on X. Let $E=\mathrm{Spec}(\mathrm{Sym}^\bullet_{\mathcal{O}_X}\mathcal{E}^\vee)$ be the associated vector bundle of rank d on X with structure morphism $\pi\colon E\to X$. The projective bundle of E is $\mathbb{P}(E)=\mathrm{Proj}(\mathrm{Sym}^\bullet_{\mathcal{O}_X}\mathcal{E}^\vee)$. We have a closed immersion $\mathbb{P}(E)\hookrightarrow P(E\oplus 1):=P(E\oplus\mathbb{A}^1_X)$ with open complementary $E\hookrightarrow\mathbb{P}(E\oplus 1)$. Let $s\colon X\to E$ be the zero section. Let $k\geqslant 0$ be an integer and $\beta\in CH_k(E)$. For any element $\bar\beta\in CH_k(\mathbb{P}(E\oplus 1))$, if the restriction of $\bar\beta$ to $CH_k(E)$ equals to β, then we have [3, Proposition 3.3]

$$
(3.2.1)\qquad s^!(\beta) = q_*(c_d(\xi)\cap\bar\beta),
$$

where $\xi = \frac{q^*(\mathcal{E}\oplus 1)}{\mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1)}$ is the universal rank d quotient bundle of $q^*(\mathcal{E}\oplus 1)$. For any element $\alpha\in CH_*(X)=\oplus_i CH_i(X)$, we denote by $\{\alpha\}_j$ the dimension j part of α, i.e., the image of α by the projection $CH_*(X)\to CH_j(X)$. Let $c(\xi)$ be the total Chern class of ξ, then we can write (3.2.1) as follows

$$
(3.2.2)\qquad s^!(\beta) = \left\{q_*(c(\xi)\cap\bar\beta)\right\}_{k-d}.
$$

By the Whitney sum formula for Chern classes [3, Theorem 3.2], we have

$$
(3.2.3)\qquad c(\xi) = c(q^*\mathcal{E})\cdot c(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1))^{-1}.
$$

Thus the formula (3.2.2) can be written in the following way

$$
\begin{aligned}
(3.2.4)\qquad s^!(\beta) &= \left\{q_*(c(q^*\mathcal{E})\cap c(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1))^{-1}\cap\bar\beta)\right\}_{k-d} \\
&= \left\{c(\mathcal{E})\cap q_*(c(\mathcal{O}_{\mathbb{P}(E\oplus 1)}(-1))^{-1}\cap\bar\beta)\right\}_{k-d}
\end{aligned}
$$

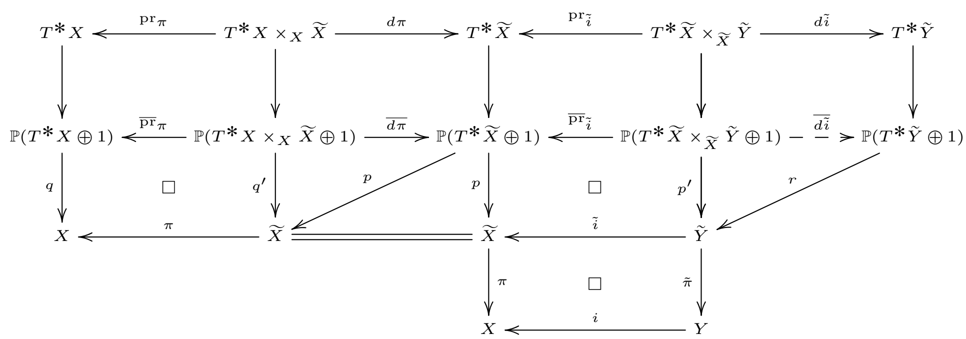
where the last equality follows from the projection formula [3, Theorem 3.2].

Lemma 3.3. *Let X and Y be smooth and connected schemes over a field k and let $i\colon Y\hookrightarrow X$ be a closed immersion of codimension c. Let $\pi\colon\widetilde{X}\to X$ be the blow up of X along Y. Let $C\subseteq T^*X$ be a conical* *closed subset purely of dimension $d = \dim X$ and let Z be a d-cycle supported on C. Suppose π and i are properly C-transversal. Then we have an equality in $CH_0(X)$:*

(3.3.1) $$\pi_*(0^!_{\widetilde{X}}(\pi^* Z)) = 0^!_X(Z) + (-1)^c \cdot (c-1) \cdot i_*(0^!_Y(i^* Z)),$$

For the definition of the Gysin map $0^!_\bullet$, see Subsection 3.1.

Proof. Let $\widetilde{Y}$ be the exceptional divisor of $\pi\colon \widetilde{X} \to X$ with projection map $\tilde{\pi}\colon \widetilde{Y} \to Y$. Let $\tilde{i}\colon \widetilde{Y} \hookrightarrow \widetilde{X}$ be the closed immersion. We have a commutative diagram

(3.3.2)


where pr_π and $\mathrm{pr}_{\tilde{i}}$ are the first projections, $d\pi\colon T^*X \times_X \widetilde{X} \to T^*\widetilde{X}$ (respectively $d\tilde{i}\colon T^*\widetilde{X} \times_{\widetilde{X}} \widetilde{Y} \to T^*\widetilde{Y}$) is the map induced by $\pi\colon \widetilde{X} \to X$ (respectively $\tilde{i}\colon \widetilde{Y} \to Y$), the maps $\overline{\mathrm{pr}}_\pi$, $\overline{d\pi}$, $\overline{\mathrm{pr}}_{\tilde{i}}$ and $\overline{d\tilde{i}}$ are the maps induced by pr_π, $d\pi$, $\mathrm{pr}_{\tilde{i}}$ and $d\tilde{i}$ respectively, all other maps are either the canonical projection morphisms or open immersions. In (3.3.2), we use the symbol "$\square$" to mean the square is a Cartesian diagram. For example, the most left-bottom square in (3.3.2) is Cartesian since q' is proper and $\mathbb{P}(T^*X \times_X \widetilde{X} \oplus 1)$ has dense image in $\mathbb{P}(T^*X \oplus 1) \times_X \widetilde{X}$. Note also that the map $\overline{d\tilde{i}}$ is only well-defined on the open subscheme $T^*\widetilde{X} \times_{\widetilde{X}} \widetilde{Y}$, but this is enough for our purpose (cf. [5, Lemma 6.4]).

For any $\alpha \in CH_d(T^*X)$, we denote by $\overline{\alpha} \in CH_d(\mathbb{P}(T^*X \oplus 1))$ an extension of α (cf. 3.2). We choose an extension $\overline{Z} \in CH_d(\mathbb{P}(T^*X \oplus 1))$ of Z. Then $\overline{\mathrm{pr}}^!_\pi(\overline{Z})$ is an extension of $\mathrm{pr}^!_\pi Z$. Since π is C-transversal, the push-forwards $d\pi_*(\mathrm{pr}^!_\pi Z)$ and $\overline{d\pi}_*(\overline{\mathrm{pr}}^!_\pi(\overline{Z}))$ are well-defined, and $\overline{d\pi}_*(\overline{\mathrm{pr}}^!_\pi(\overline{Z}))$ is an extension of $d\pi_*(\mathrm{pr}^!_\pi Z)$.

Since $\tilde{\pi}$ is smooth, thus $\tilde{\pi}$ is $i^\circ C$-transversal by [5, Lemma 3.4.1]. By Lemma 1.3, $\tilde{i}$ is $\pi^\circ C$-transversal and we have

(3.3.3) $$\tilde{\pi}^* i^* Z = \tilde{i}^* \pi^* Z.$$

The following exact sequence ($\tilde{i}_*\Omega^1_{\widetilde{Y}/Y} \simeq \Omega^1_{\widetilde{X}/X}$)

(3.3.4) $$0 \to \pi^*\Omega^1_X \to \Omega^1_{\widetilde{X}} \to \tilde{i}_*\Omega^1_{\widetilde{Y}/Y} \to 0$$

gives a resolution of $\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}$ by locally free sheaves of finite rank. Thus the localized Chern class $c_{k}{}^X_Y(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) \in CH^*(Y \to X)$ is well-defined for $k \geqslant 1$ (cf. Subsection 2.1). In order to simplify the notation, we put $c_k^{\mathrm{loc}}(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) := c_{k}{}^X_Y(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y})$ and $c_k^{\mathrm{loc}}(p^*\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) := c_{k}{}^{\mathbb{P}(T^*\widetilde{X}\oplus 1)}_{\mathbb{P}(T^*\widetilde{X}\times_{\widetilde{X}}\widetilde{Y}\oplus 1)}(p^*\tilde{i}_*\Omega^1_{\widetilde{Y}/Y})$. Similarly, we denote by c^{loc} the total localized Chern class. Applying the Whitney sum formula for (localized) Chern classes (cf. [1, Proposition 3.1]) to the exact sequence (3.3.4), we get

(3.3.5) $$c(\Omega^1_{\widetilde{X}}) = c(\pi^*\Omega^1_X) \cdot c^{\mathrm{loc}}(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) = c(\pi^*\Omega^1_X) + c(\pi^*\Omega^1_X) \cdot (c^{\mathrm{loc}}(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y}) - 1).$$

We will simply denote by $\mathcal{O}(1)$ for $\mathcal{O}_{\mathbb{P}(T^*\widetilde{X}\oplus 1)}(1)$ (and also for $\mathcal{O}_{\mathbb{P}(T^*X\oplus 1)}(1)$ and so on) in the following calculations. We have

$$
\begin{aligned}
(3.3.6)\qquad \pi_*(0^!_{\widetilde{X}}(\pi^*Z)) &\overset{(3.2.4)}{=} \left\{\pi_*\left(c(\Omega^1_{\widetilde{X}})\cap p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right)\right\}_0\\
&\overset{(3.3.5)}{=} \left\{\pi_*\left(c(\pi^*\Omega^1_X)\cdot c^{\mathrm{loc}}(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y})\cap p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right)\right\}_0\\
&\overset{(a)}{=} \left\{c(\Omega^1_X)\cap \pi_*\left(c^{\mathrm{loc}}(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y})\cap p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right)\right\}_0.\\
&= \left\{c(\Omega^1_X)\cap \pi_*\left(p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right)\right\}_0\\
&\quad + \left\{c(\Omega^1_X)\cap \pi_*\left(\left(c^{\mathrm{loc}}(\tilde{i}_*\Omega^1_{\widetilde{Y}/Y})-1\right)\cap p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right)\right\}_0\\
(3.3.7)\qquad &\overset{lem. 2.4}{=} \left\{c(\Omega^1_X)\cap \pi_*\left(p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right)\right\}_0\\
&+ \left\{i_*\left(c(i^*\Omega^1_X)\cap c(N_{Y/X})^{-1}\cap \tilde{\pi}_*\left(\Phi(X,Y)\cap \tilde{i}^!\left(p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right)\right)\right)\right\}_0
\end{aligned}
$$

where (a) follows from the projection formula [3, Theorem 3.2].

We calculate the first term of (3.3.7). Using projection formula [3, Theorem 3.2] and [3, Proposition 17.3.2], we have

$$
\begin{aligned}
(3.3.8)\qquad &\left\{c(\Omega^1_X)\cap \pi_*\left(p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right)\right\}_0\\
&= \left\{c(\Omega^1_X)\cap \pi_*p_*\overline{d\pi}_*(c(\overline{d\pi}^*\mathcal{O}(-1))^{-1}\cap \overline{\mathrm{pr}}^!_{\pi}\overline{Z})\right\}_0\\
&= \left\{c(\Omega^1_X)\cap \pi_*q'_*(c(\mathcal{O}(-1))^{-1}\cap \overline{\mathrm{pr}}^!_{\pi}\overline{Z})\right\}_0\\
&= \left\{c(\Omega^1_X)\cap \pi_*q'_*\left(c(\overline{\mathrm{pr}}^*_{\pi}\mathcal{O}(-1))^{-1}\cap \overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right\}_0\\
&= \left\{c(\Omega^1_X)\cap \pi_*q'_*\overline{\mathrm{pr}}^!_{\pi}\left(c(\mathcal{O}(-1))^{-1}\cap \overline{Z}\right)\right\}_0\\
&= \left\{c(\Omega^1_X)\cap \pi_*\pi^*q'_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{Z}\right)\right\}_0\\
&\overset{(b)}{=} \left\{c(\Omega^1_X)\cap q'_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{Z}\right)\right\}_0\\
&\overset{(3.2.4)}{=} 0^!_X(Z)
\end{aligned}
$$

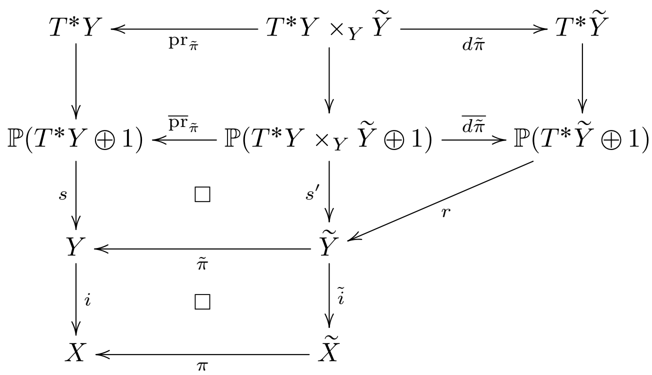
where (b) follows from [3, Proposition 6.7] since π is a blow-up.

Now we calculate the second term of (3.3.7). First, by push-forward [3, Theorem 6.2(a)] and the projection formula [3, Theorem 3.2], we have

$$
\begin{aligned}
(3.3.9)\qquad &\tilde{i}^!\left(p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right) = p'_*\overline{\mathrm{pr}}^*_{\tilde{i}}\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\\
&= p'_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{\mathrm{pr}}^!_{\tilde{i}}\overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right) = r_*\overline{d\tilde{i}}_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{\mathrm{pr}}^!_{\tilde{i}}\overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\\
&= r_*\overline{d\tilde{i}}_*\left(c(\overline{d\tilde{i}}^*\mathcal{O}(-1))^{-1}\cap \overline{\mathrm{pr}}^!_{\tilde{i}}\overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right) = r_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\tilde{i}}_*\overline{\mathrm{pr}}^!_{\tilde{i}}\overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\\
&= r_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{\tilde{i}^*\pi^*Z}\right)\\
&\overset{(3.3.3)}{=} r_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{\tilde{\pi}^*i^*Z}\right).
\end{aligned}
$$

Consider the following commutative diagram induced from the morphism $\tilde{\pi}\colon \widetilde{Y} \to Y$.

(3.3.10)



Since $\overline{d\tilde{\pi}}_*\overline{\mathrm{pr}}^!_{\pi}\overline{i^*Z}$ is an extension of $\tilde{\pi}^*i^*Z$ to $\mathbb{P}(T^*\widetilde{Y}\oplus 1)$, thus (3.3.9) equals to

$$
\begin{aligned}
(3.3.11)\qquad \tilde{i}^!\left(p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right) &= r_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\tilde{\pi}}_*\overline{\mathrm{pr}}^!_{\tilde{\pi}}\overline{i^*Z}\right)\\
&\overset{(c)}{=} r_*\overline{d\tilde{\pi}}_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{\mathrm{pr}}^!_{\tilde{\pi}}\overline{i^*Z}\right) = r_*\overline{d\tilde{\pi}}_*\left(\overline{\mathrm{pr}}^*_{\pi}c(\mathcal{O}(-1))^{-1}\cap \overline{\mathrm{pr}}^!_{\tilde{\pi}}\overline{i^*Z}\right)\\
&\overset{(d)}{=} r_*\overline{d\tilde{\pi}}_*\overline{\mathrm{pr}}^!_{\tilde{\pi}}\left(c(\mathcal{O}(-1))^{-1}\cap \overline{i^*Z}\right)
\end{aligned}
$$

where we used the projection formula [3, Theorem 3.2] in step (c), and (d) follows from [3, Proposition 17.3.2]. By the commutative diagram (3.3.10) and the push-forward formula [3, Theorem 6.2], we have

$$
(3.3.12)\qquad r_*\overline{d\tilde{\pi}}_*\overline{\mathrm{pr}}^!_{\tilde{\pi}} = s'_*\overline{\mathrm{pr}}^!_{\tilde{\pi}} = \tilde{\pi}^!s_* = \tilde{\pi}^*s_*.
$$

By (3.3.11) and (3.3.12), the second term of (3.3.7) equals to

$$
\begin{aligned}
(3.3.13)\qquad &\left\{i_*\left(c(i^*\Omega^1_X)\cap c(N_{Y/X})^{-1}\cap \tilde{\pi}_*\left(\Phi(X,Y)\cap \tilde{i}^!\left(p_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{d\pi}_*\overline{\mathrm{pr}}^!_{\pi}\overline{Z}\right)\right)\right)\right)\right\}_0\\
&= \left\{i_*\left(c(i^*\Omega^1_X)\cap c(N_{Y/X})^{-1}\cap \tilde{\pi}_*\left(\Phi(X,Y)\cap \tilde{\pi}^*s_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{i^*Z}\right)\right)\right)\right\}_0\\
&\overset{(2.4.3)}{=}(-1)^c\cdot(c-1)\left\{i_*\left(c(i^*\Omega^1_X)\cap c(N_{Y/X})^{-1}\cap s_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{i^*Z}\right)\right)\right\}_0\\
&\overset{(1)}{=}(-1)^c\cdot(c-1)\left\{i_*\left(c(\Omega^1_Y)\cap s_*\left(c(\mathcal{O}(-1))^{-1}\cap \overline{i^*Z}\right)\right)\right\}_0\\
&\overset{(3.2.4)}{=}(-1)^c\cdot(c-1)\cdot i_*0^!_Y(i^*Z).
\end{aligned}
$$

where the step (1) follows from $c(i^*\Omega^1_X)\cdot c(N_{Y/X})^{-1} = c(\Omega^1_Y)$ since we have an exact sequence

$$
(3.3.14)\qquad 0 \to N_{Y/X} \to i^*\Omega^1_X \to \Omega^1_Y \to 0,
$$

where $N_{Y/X}$ is the conormal sheaf associated to the the regular immersion $i\colon Y \to X$.

Finally, by (3.3.6), (3.3.8) and (3.3.12), we get (3.3.1). This finishes the proof. □

References

[1] A. Abbes, *Cycles on arithmetic surfaces*, Compos. Math., 122 (2000): 23-111.

[2] A. Beilinson, *Constructible sheaves are holonomic*, Selecta Mathematica, 22, Issue 4 (2016):1797-1819.

[3] W. Fulton, *Intersection theory*, 2nd edition, Springer New York (1998).

[4] K. Kato and T. Saito, *On the conductor formula of Bloch*, Publications Mathématiques de l'IHÉS 100 (2004): 5-151.

[5] T. Saito, *The characteristic cycle and the singular support of a constructible sheaf*, Inventiones mathematicae, 207 (2017): 597-695.

[6] N. Umezaki, E. Yang and Y. Zhao, *Characteristic class and the ε-factor of an étale sheaf*, preprint.

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