第二章、跳过程

§2.1 泊松过程

§2.2 跳过程的定义(简略)、转移概率(略)

§2.3 常返(略)

§2.4 正常返与不变分布(简略)

§2.5 逆过程与可逆分布(略)

~~§2.6 连续时间马氏链~~

- 回顾：马氏链的定义. S 非空、可数
- 离散时间参数：$\{X_n : n \geqslant 0\}$, $n \in \mathbb{Z}_+$.

$$P(X_{n+1} = j | X_n = i, X_0 = i_0, \cdots, X_{n-1} = i_{n-1}) = p_{ij},$$

- 连续时间参数. $\{X_t : t \geqslant 0\}$, $t \in [0, \infty)$.
- 定义2.0.1. $\forall n$, $0 \leqslant t_1 < \cdots < t_n < s < t+s$, 及其$i_1, \cdots, i_n, i, j \in S$,

$$P(X_{t+s} = j | X_s = i, X_{t_1} = i_1, \cdots, X_{t_n} = i_n) = P(X_{t+s} = j | X_s = i),$$

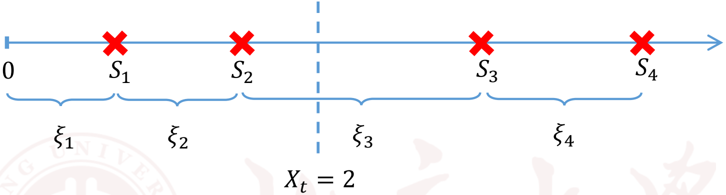
则称$\{X_t\}$ 是(连续时间参数的)马氏链.

- 时齐：$\star\star$ 不依赖于t.

转移概率：$p_{ij}(t) := \star\star$; 转移矩阵：$\mathbf{P}(t) := (p_{ij}(t))_{i,j \in S}$.

§2.1 泊松过程(Poisson Process)

一、基于指数分布



- $\xi_1, \xi_2, \ldots$ i.i.d. $\sim \mathrm{Exp}(\lambda)$. $P(\xi > t) = e^{-\lambda t}$.
- $S_n := \xi_1 + \cdots + \xi_n, \quad S_0 := 0.$
- $X_t := |\{n \geqslant 1 : S_n \leqslant t\}| = \sup\{n \geqslant 0 : S_n \leqslant t\}.$
- 定义2.1.1. 泊松过程: $\{X_t : t \geqslant 0\}$.

  等待时间: $\xi_1, \xi_2, \cdots$; 到达时刻/跳跃时间: $S_1, S_2, \cdots$

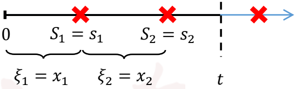
  泊松流: $\{S_n : n \geqslant 1\} = \{S_1, S_2, \ldots\}$.

  速率/参数/强度: λ.

定理 (定理2.1.3)

X_t 服从参数为λt 的泊松分布.

- 固定t. $P(X_t=0)=P(S_1>t)=e^{-\lambda t}$.
- $\forall k\geqslant 1$. $\{X_t=k\}=\{S_k\leqslant t< S_{k+1}\}$.
- $s_r=x_1+\cdots+x_r$,



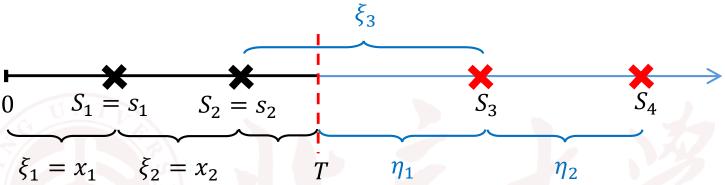
$$\begin{aligned} C_k &:= \{\vec{x}\in\mathbb{R}^k: x_1,\cdots,x_k\geqslant 0,\ 且\ s_k\leqslant t\},\\ D_k &:= \{(\vec{x},x_{k+1}): \vec{x}\in C_k,\ 且\ x_{k+1}>t-s_k\}. \end{aligned}$$

- $\{S_k\leqslant t<S_{k+1}\}=\{(\xi_1,\cdots,\xi_k,\xi_{k+1})\in D_k\}$.

$$\begin{aligned} P(\star) &= \int_{(\vec{x},x_{k+1})\in D_k}\lambda^{k+1}e^{-\lambda s_{k+1}}\mathrm{d}\vec{x}\mathrm{d}x_{k+1}\\ &= \int_{\vec{x}\in C_k}\lambda^k e^{-\lambda s_k}e^{-\lambda(t-s_k)}\mathrm{d}\vec{x}=\lambda^k e^{-\lambda t}\times\frac{t^k}{k!}. \end{aligned}$$

定理 (定理2.1.4)

固定T. $X_T, \eta_1, \eta_2, \cdots$ 相互独立, $\eta_n \sim \mathrm{Exp}(\lambda)$.



- 将$\{X_T = k, \eta_1 > t_1, \eta_2 > t_2 \cdots, \eta_n > t_n\}$ 改写为

$$\{S_k \leqslant T, S_{k+1} > T + t_1, \xi_{k+2} > t_2, \cdots, \xi_{k+n} > t_n\}.$$

- 还需验证$P(\star\star) = \frac{(\lambda T)^k}{k!} e^{-\lambda T} \cdot e^{-\lambda t_1}$.

$$P(\star\star) = P(S_k \leqslant T, S_{k+1} > T + t_1) \times e^{-\lambda(t_2+\cdots+t_n)}.$$

- 往验证$P(S_k \leqslant T, S_{k+1} > T + t_1) = \frac{(\lambda T)^k}{k!} e^{-\lambda T} \cdot e^{-\lambda t_1}$.
- $s_r = x_1 + \cdots + x_r$,

$$C_k := \{\vec{x} \in \mathbb{R}^k : x_1, \cdots, x_k \geqslant 0, \text{ 且 } s_k \leqslant T\},$$
$$\hat{D}_k := \{(\vec{x}, x_{k+1}) : \vec{x} \in C_k, \text{ 且 } x_{k+1} > (T - s_k) + t_1\}.$$

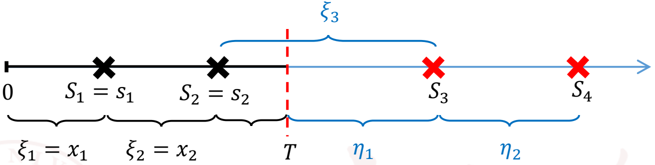
- $\{S_k \leqslant T, S_{k+1} > T + t_1\} = \{(\xi_1, \cdots, \xi_k, \xi_{k+1}) \in \hat{D}_k\}$.

$$P(\star) = \int_{(\vec{x}, x_{k+1}) \in \hat{D}_k} \lambda^{k+1} e^{-\lambda s_{k+1}} \mathrm{d}\vec{x} \mathrm{d}x_{k+1}$$
$$= \int_{\vec{x} \in C_k} \lambda^k e^{-\lambda s_k} e^{-\lambda(T - s_k + t_1)} \mathrm{d}\vec{x} = \lambda^k e^{-\lambda(T + t_1)} \times \frac{T^k}{k!}.$$

- 注：指数分布的无记忆性. $\{S_k \leqslant T < S_{k+1}, S_{k+1} > T + t_1\}$,

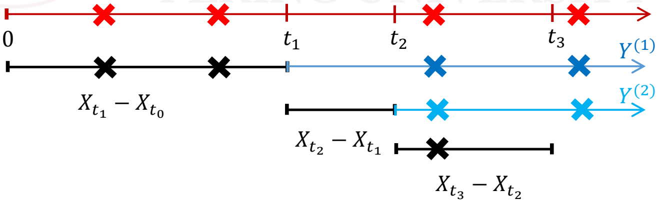
$$P(\xi_{k+1} > (T - s_k) + t_1 | \xi_{k+1} > (T - s_k)).$$

- 推论2.1.5 给定$T > 0$. $\{Y_t := X_{T+t} - X_T\}$ 是速率为λ 的泊松过程, 且与X_T 独立.



- 推论2.1.6. $\forall r \geqslant 1$, $0 = t_0 < t_1 < \cdots < t_r$, $\forall k_1, \cdots, k_r$,

$$P(X_{t_i} - X_{t_{i-1}} = k_i,\ i = 1, \cdots, r) = \prod_{i=1}^{r} P(X_{t_i - t_{i-1}} = k_i).$$



- $\forall r \geqslant 1,\ 0 = t_0 < t_1 < \cdots < t_r,\ \forall k_1, \cdots, k_r,$

$$P(X_{t_i} - X_{t_{i-1}} = k_i,\ i = 1, \cdots, r) = \prod_{i=1}^{r} P(X_{t_i - t_{i-1}} = k_i).$$

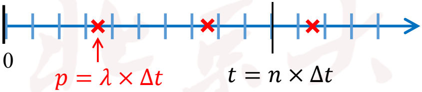
- 推论2.1.8(马氏性). $\forall r \geqslant 1,\ 0 \leqslant t_1 < \cdots < t_r < t < t+s,$ $0 \leqslant k_1 \leqslant \cdots \leqslant k_r \leqslant k \leqslant \ell,$

$$\begin{aligned} & P\big(X_{t+s} = \ell \big| X_t = k, X_{t_1} = k_1, \cdots, X_{t_r} = k_r\big) \\ = \ & P(X_{t+s} = \ell | X_t = k) = P(X_s = \ell - k). \end{aligned}$$

- 注: 泊松过程是独立、平稳增量过程. (定义2.1.17)

二、基于均匀分布与泊松分布

- 泊松流: $\{S_n : n \geqslant 1\} = \{S_1, S_2, \cdots\} \triangleq \Xi$.
- 随机点集: $\Xi(\omega) = \{S_1(\omega), S_2(\omega), \cdots\} \subseteq \mathbb{R}_+$.
- $X_T(\omega) = |[0, T] \cap \Xi(\omega)|$.
- 例. 放射性物质放射粒子的过程, $\{X_t\}$.

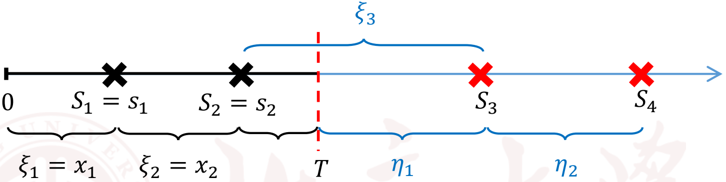


- 命题2.1.9. $P(S_1 \leqslant s | X_T = 1) = s/T, \forall s < T$.
- 证: $\{S_1 \leqslant s\}$ vs $\{S_1 \leqslant T < S_2\}$.
- 命题2.1.10. 在$\{X_T = k\}$ 上, $(S_1, \cdots, S_k)$ 的条件密度为

$$\frac{k!}{T^k} \times \mathbf{1}_{\{0<s_1<\cdots<s_k<T\}}.$$

- 注: k 个i.i.d. $U(0,1)$ 随机变量的顺序统计量的联合密度.

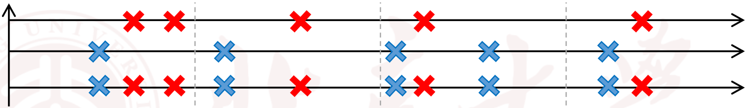
- 注: $\Xi_1 = \Xi \cap [0,1] \overset{d}{=} \{U_1, \cdots, U_Z\}$, 其中$Z \sim P(\lambda)$, U_i's $\sim U(0,1)$, 全部独立. (定理2.1.3, 命题2.1.9 & 2.1.10).



- 注: Ξ_1 与$\Xi \cap [1,\infty)$ 独立, (vs定理2.1.4 & 推论2.1.5.).
- 命题2.1.11(泊松流的构造). Z_n's, $U_{n,i}$'s 独立, $Z_n \sim P(\lambda)$, $U_{n,i} \sim U(n-1,n)$. 则$\bigcup_{n=1}^{\infty}\{U_{n,1}, \cdots, U_{n,Z_n}\}$ 是泊松流.

三、泊松过程/泊松流的合并与细分

- 定理2.1.12. 泊松流与泊松过程的合并.
  (1) 设Ξ, $\hat{\Xi}$ 独立, 速率: λ_1, λ_2. 则$\Xi \cup \hat{\Xi}$ 是⋆, 速率为$\lambda_1 + \lambda_2$.
  (2) 设$\{X_t\}$, $\{Y_t\}$ 独立, ⋆. 则$\{X_t + Y_t\}$ 是⋆, ⋆.



- 证: 设 Ξ: Ξ_n's, i.i.d., $\Xi_1 = \{U_1, \cdots, U_Z\}$;
  $\hat{\Xi}$: $\hat{\Xi}_n$'s, i.i.d., $\hat{\Xi}_1 = \{\tilde{U}_1, \cdots, \tilde{U}_W\}$; 全部独立.
  则 $\Xi \cup \hat{\Xi}$: $(\Xi_n \cup \hat{\Xi}_n)$'s, i.i.d., $\Xi_1 \cup \hat{\Xi}_1 \stackrel{d}{=} \{U_1, \cdots, U_{Z+W}\}$.
- 定理2.1.13. 假设Ξ 是速率为λ 的泊松流. 则$\tilde{\Xi}$ 与$\hat{\Xi}$ 是相互独立的泊松流, 速率分别为λp 与$\lambda(1-p)$.
- 注: “大”泊松流= 独立的“小”泊松流.

例2.1.15 (复合泊松过程). $\{X_t\}$ 是泊松过程, 速率为λ. $\phi_1, \phi_2, \cdots$ i.i.d. 且与$\{X_t\}$ 独立. 复合泊松过程: $\{Y_t = \sum_{n=1}^{X_t} \phi_n\}$.

- 譬如, 旅行团, 第n 个有ϕ_n 个人.
  Y_t: 时刻t 之前到达的顾客总数.
- 按时间发展累计; 或按旅行团中的顾客数进行分类累计:

$$Y_t = \sum_{k=1}^{\infty} k X_t^{(k)}, \quad 其中\ X_t^{(k)} = \sum_{n=1}^{X_t} \mathbf{1}_{\{\phi_n = k\}}.$$

- 譬如, 只关心“至少有50 人的旅行团”,
  $\hat{X}_t :=$ 时刻t 之前到达的团数$= \sum_{k=50}^{\infty} X_t^{(k)}$, 为泊松过程,
  $\hat{Y}_t :=$ 时刻t 之前到达的顾客总数$= \sum_{k=50}^{\infty} k X_t^{(k)}$.

四、总结与拓展

- 泊松流的结构: 指数分布、泊松分布& 均匀分布.
- 泊松流不同时间区间的独立性、平稳性;
  泊松过程的独立、平稳增量性.
- 泊松过程的马氏性, 转移概率.
- 泊松流的合并与(靠色子)细分:

“大”闹钟 $\rightleftarrows$ 有限或可列个独立的“小”闹钟.

一、独立指数闹钟模型

- $\forall i \in S$, 放置大闹钟, 速率为$q_i \geqslant 0$; 色子$\{\hat{p}_{ij} : j \neq i\}$.
- 假设: 若$q_i = 0$ (吸收态), 则 $\hat{p}_{ii} := 1$;

  若$q_i > 0$, 则 $\hat{p}_{ii} := 0$, $\sum_{j\neq i} \hat{p}_{ij} = 1$.
- 大闹钟细分为一组小闹钟:

$$q_{ij} := q_i\hat{p}_{ij}, \ \forall j \neq i; \quad \text{补充定义 } q_{ii} := -q_i.$$

- $\mathbf{Q} = (q_{ij})_{S\times S}$ 为速率矩阵(定义2.2.1):

$$q_{ij} \geqslant 0, \ \forall j \neq i; \quad \sum_{j\in S} q_{ij} = 0, \ \forall i \in S.$$

- 反过来, 一组小闹钟可合并为大闹钟.

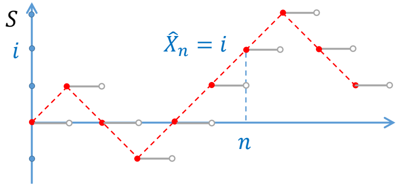
$$q_i = \sum_{j\neq i} q_{ij} < \infty, \quad \hat{p}_{ij} = \begin{cases} q_{ij}/q_i, & q_i > 0, j \neq i; \\ \mathbf{1}_{\{j=i\}}, & q_i = 0. \end{cases}$$

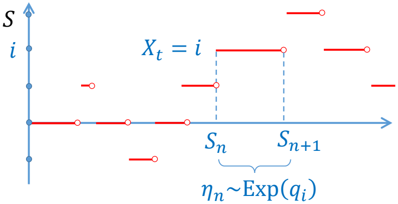
二、跳过程的定义

- 嵌入链$\{\hat{X}_n\}$: 以$\hat{\mathbf{P}} = (\hat{p}_{ij})_{S\times S}$ 为转移矩阵的马氏链.
- 闹钟: 拉伸时间.

  设$\xi_0, \xi_1, \xi_2, \ldots$ i.i.d. $\sim \text{Exp}(1)$, 与$\{\hat{X}_n\}$ 独立.
- 注: $\xi \sim \text{Exp}(1)$, 则$\eta = \xi/q \sim \text{Exp}(q)$.
- 等待时间: $\eta_n = \dfrac{\xi_n}{q_{\hat{X}_n}}$, $n = 0, 1, 2, \cdots$.

$$\eta_0 = \frac{\xi_0}{q_{\hat{X}_0}}, \quad \eta_1 = \frac{\xi_1}{q_{\hat{X}_1}}, \quad \eta_2 = \frac{\xi_2}{q_{\hat{X}_2}}, \ldots$$





- 注: 在$\{\hat{X}_n\} = \vec{i}$ 的条件下,

$$\eta_0, \eta_1, \eta_2, \ldots \text{ 相互独立}, \quad \eta_n = \frac{\xi_n}{q_{i_n}} \sim \mathrm{Exp}(q_{i_n}).$$

- 跳跃时刻: 例. $S_1 = \eta_0$ 为第一次跳跃时刻.

$$S_0 := 0; \qquad S_n = \eta_0 + \eta_1 + \cdots + \eta_{n-1}, \quad n = 1, 2, \cdots$$

- 分段定义轨道:

$$X|_{[S_0,S_1)} := \hat{X}_0, \quad X|_{[S_1,S_2)} := \hat{X}_1, \ \ldots.$$

- 大前提:

$$P\left(\lim_{n\to\infty} S_n = \infty \middle| \hat{X}_0 = i\right) = 1, \quad \forall i \in S.$$

- 注: 充分条件: $\{\hat{X}_n\}$ 常返. (引理2.3.1, 推论2.3.2).
  另有充分条件: 引理2.2.2, (命题2.2.5及其)推论2.2.6.

三、例子(建模，滞后)

四、转移概率与转移速率

- $p_{ij}(t) = P_i(X_t = j)$: $n \geqslant 0$, $i_0 = i$, $i_n = j$.

$$P_i(\hat{X}_1 = i_1, \ldots, \hat{X}_n = i_n; S_n \leqslant t < S_{n+1}).$$

- 命题2.2.13 (马氏性).$0 \leqslant t_1 < \cdots < t_n < t$, $s > 0$,

$$P(X_{t+s} = j | X_t = i, X_{t_1} = i_1, \cdots, X_{t_n} = i_n) = p_{ij}(s).$$

- 推论2.2.14 (Chapman-Kolmogorov 等式).

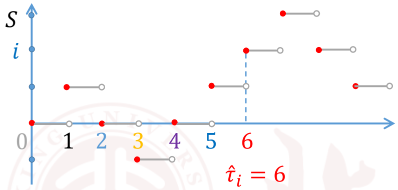
$$\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s), \quad \text{即 } p_{ij}(t+s) = \sum_k p_{ik}(t)p_{kj}(s).$$

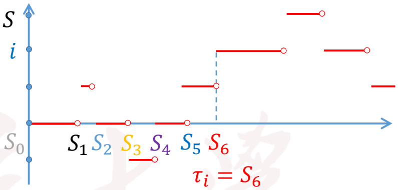
- 定理2.2.15 (连续可微性). $\mathbf{P}'(0) = \mathbf{Q}$, 即$p'_{ij}(0) = q_{ij}$, $\forall i, j$.
- 命题2.2.15. Kolmogorov 前进、后退方程:

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q} = \mathbf{Q}\mathbf{P}(t), \text{ 即 } p'_{ij}(t) = \sum_{k \in S} p_{ik}(t)q_{kj} = \sum_{k \in S} q_{ik}p_{kj}(t).$$

五、首达时

- $\tau_D := \inf\{t \geqslant 0 : X_t \in D\}$. $\tau_{\{i\}} \triangleq \tau_i$.





- 嵌入链的首达时记为$\hat{\tau}_D$. 则

$$\tau_D = S_{\hat{\tau}_D}, \text{ 若 } \hat{\tau}_D < \infty; \quad \tau_D = \infty, \text{ 若 } \hat{\tau}_D = \infty.$$

- 注: 类似地, $X_0 = i$ 时, $\tau_i = 0 < \sigma_i$.

$$\sigma_D := \inf\{t \geqslant S_1 : X_t \in D\} = S_{\hat{\sigma}_D} \text{ 或 } \infty.$$

- $x_i = P_i(\hat{\tau}_D < \infty)$, $y_i := E_i\tau_D$ 的方程组.

§2.3 常返

- 可达、互通、不可约、常返/非常返: 与嵌入链相同!
- 常返: $P_i(\forall t > 0,\ \exists s > t,\ \text{s.t. } X_s = i) = 1$.
  否则非常返(暂态). (定义2.3.4)
- 格林函数: $\zeta_1, \zeta_2, \cdots$ i.i.d., $\sim \mathrm{Exp}(q_i)$.

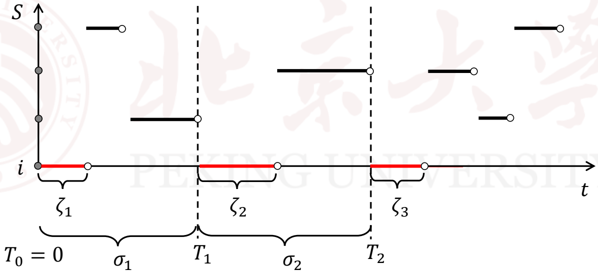
$$V_i := \int_0^\infty \mathbf{1}_{\{X_t=i\}} \mathrm{d}t = \begin{cases} \zeta_1 + \zeta_2 + \cdots, & \text{若 } \hat{V}_i = \infty, \\ \zeta_1 + \cdots + \zeta_r, & \text{若 } \hat{V}_i = r. \end{cases}$$

$$G_{ij} := E_i V_j = \int_0^\infty p_{ij}(t)\mathrm{d}t = \frac{1}{q_j}\hat{G}_{ij}.$$

- 命题2.3.4. i 常返iff i 是$\hat{\mathbf{P}}$ 的常返态iff $G_{ii} = \infty$.
- 注: 常返类、常返马氏链.

§2.4. 正常返与不变分布

- 假设不可约、常返, $|S|>1$.
- 引理2.4.1. 设$X_0=i$. $(\sigma_{i,1},\zeta_{i,1}),(\sigma_{i,2},\zeta_{i,2}),\cdots$ i.i.d..

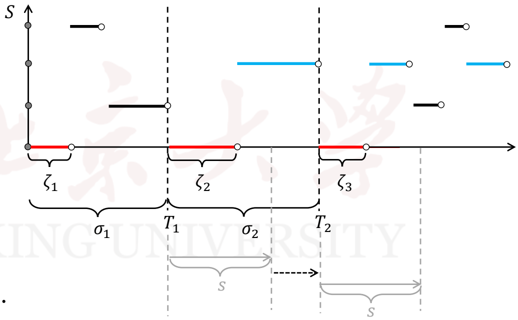


- 注: 事实上, $X|_{[T_0,T_1)},X|_{[T_1,T_2)},\cdots$ i.i.d., 游弋.

命题 (访问频率, 命题2.4.2)

$$P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t 1_{\{X_s=i\}}\mathrm{d}s=\frac{1}{q_iE_i\sigma_i}\right)=1.$$

- 证: $\star\approx\frac{S_r}{T_r}\approx\frac{E\zeta_1}{E_i\sigma_i}$.
- 正常返: $E_i\sigma_i<\infty$;
  vs 零常返. (定义2.4.3)
- 另, $\star\approx\frac{E_o\int_0^{\sigma_o}\mathbf{1}_{\{X_t=i\}}\mathrm{d}t}{E_o\sigma_o}$.



- 命题2.4.6(& 引理2.4.5).
  假设不可约、常返. 则μ 为不变测度: $\forall s>0$,

$$\sum_{j\in S}\mu_jp_{ji}(s)=\mu_i,\quad i\in S.$$

- 定理2.4.8. 设不可约. 则下面三条等价:
  (1) 所有状态正常返, (2) 存在正常返态, (3) 存在不变分布.
  (定义2.4.7. $\pi\mathbf{P}(t) = \pi, \forall t.$) 特别地,

$$\pi_i = \frac{1}{q_i E_i \sigma_i} = \frac{1}{E_o \sigma_o} E_o \int_0^{\sigma_o} \mathbf{1}_{\{X_t = i\}} \mathrm{d}t.$$

- 推论2.4.9. 设不可约、常返. 则
  若π 是S 上的测度(分布), 则$\pi\mathbf{Q} = 0$ iff $\pi_i \propto \mu_i$ (不变分布).
- 证: $\pi\mathbf{Q} = 0$ iff

$$\sum_j \pi_j q_{ji} = 0, \forall i \Leftrightarrow \sum_{j \neq i} \pi_j q_j \hat{p}_{ji} = \pi_i q_i, \forall i \Leftrightarrow \tilde{\pi}\hat{\mathbf{P}} = \tilde{\pi}.$$

- $\hat{\mu}$ 是$\hat{\mu}\hat{\mathbf{P}} = \hat{\mu}$ 的 "唯一" 解.

$$\hat{\mu}_i := E_o \sum_{n=0}^{\hat{\sigma}_o - 1} \mathbf{1}_{\{\hat{X}_n = j\}}. \quad \mu_i = \frac{1}{q_i} \cdot \hat{\mu}_i, \text{ 即 } \mu_i q_i = \hat{\mu}_i.$$

- $\pi\mathbf{Q} = 0$ iff $\pi_i q_i \propto \mu_i q_i$, 即$\pi_i \propto \mu_i$.

例2.4.11. 生灭过程, $S=\mathbb{Z}_+$. $\beta_i=q_ib_i$; $\delta_i=q_id_i$. $(b_0=1)$

- (1) 带反射边界的SRW. 取$b_i=d_i=1/2$, $i\geqslant 1$.
- $\{\hat{X}_n\}$ 零常返. 唯一的不变测度: $\mu_0=1/2$; $\mu_i\equiv 1$, $i\geqslant 1$.
- 解方程$\pi\mathbf{Q}=0$:

$$\pi_0\beta_0=\pi_1\delta_1,\quad \pi_1(\beta_1+\delta_1)=\pi_0\beta_0+\pi_2\delta_2,\cdots$$

$$\text{即},\quad \pi_0\beta_0=\pi_1\delta_1,\quad \pi_1\beta_1=\pi_2\delta_2,$$

- 得到$\{X_t\}$ 的不变分布形如:

$$\pi_i=\pi_0\frac{\beta_0\cdots\beta_{i-1}}{\delta_1\cdots\delta_i}=\pi_0\frac{b_0\cdots b_{i-1}}{d_1\cdots d_i}\times\frac{q_0}{q_i}=\pi_0\times\frac{1}{2}\times\frac{q_0}{q_i}.$$

- 取$q_0=1$, $q_i=i^2$, $i\geqslant 1$ 使得可归一化. 于是, $\{X_t\}$ 正常返.
- (2) 任取正常返的$\{\hat{X}_n\}$, 不变分布记为$\hat{\mu}$.
- 令$q_i=\hat{\mu}_i$, $\forall i$. 则$\pi\mathbf{Q}=0$ iff $\pi_i\propto\mu_i=\hat{\mu}_i/q_i=1$, 不可归一化, 故$\{X_t\}$ 零常返.

总结:

- 正常返: $E_i\sigma_i < \infty$.
- 正常返iff 存在不变分布π. 此时, $\pi_i = \frac{1}{q_i E_i \sigma_i}$.
- 设常返. 解(等价)方程$\pi\mathbf{Q} = 0$, $\hat{\pi}\hat{\mathbf{P}} = \hat{\pi}$. “唯一”解.
- 概率流速率: $\sum_{j\neq i} \pi_j q_{ji} = \pi_i q_i$.
- 遍历定理(§2.4 习题3 $\sim$ 5):

$$\sum_{i\in S}\left|\frac{1}{t}\int_0^t \mathbf{1}_{\{X_s=i\}}\mathrm{d}s - \pi_i\right| \to 0,\ \lim_{t\to\infty}\frac{1}{t}\int_0^t f(X_s)\mathrm{d}s \to \sum_{i\in S}\pi_i f(i),\ \text{a.s.}.$$

- 强遍历(§2.4 习题6):

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j, \quad \forall j.$$

§2.5 逆过程与可逆分布

- 假设**Q** 不可约、正常返. π 为不变分布. 令

$$\tilde{q}_{ij} = \frac{\pi_j q_{ji}}{\pi_i}, \quad \forall i, j \in S; \qquad \tilde{\mathbf{Q}} = (\tilde{q}_{ij})_{S\times S}.$$

- $\pi\mathbf{Q} = 0$, 即$\pi_i q_i = \sum_{j\neq i} \pi_j q_{ji} = \sum_{j\neq i} \pi_j q_j \hat{p}_{ji}$.
- 故, $\tilde{\mathbf{Q}}$ 仍为速率矩阵, 且$\tilde{q}_i = q_i$.

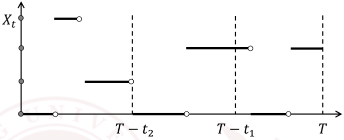
$$\sum_{j\neq i} \tilde{q}_{ij} = \sum_{j\neq i} \frac{\pi_j q_{ji}}{\pi_i} = q_i = -q_{ii} = -\tilde{q}_{ii}.$$

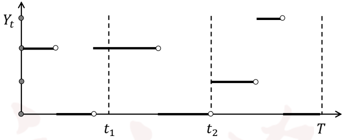
- $\tilde{\mathbf{Q}}$ 对应的嵌入链: $\forall j \neq i$,

$$\hat{q}_{ij} := \frac{\tilde{q}_{ij}}{q_i} = \frac{\pi_j q_j \hat{p}_{ji}}{\pi_i q_i} = \frac{\hat{\pi}_j \hat{p}_{ji}}{\hat{\pi}_i}.$$

- 注: 若$\hat{\pi}$ 可归一化, 则$\{\hat{X}_n\}$ 的逆过程以$(\hat{q}_{ij})$ 为转移矩阵.

- $\tilde{q}_{ij} = \frac{\pi_j q_{ji}}{\pi_i}$. $\tilde{\mathbf{Q}}$ 的闹钟速率为$\tilde{q}_i = q_i$, 色子为$\hat{q}_{ij} = \hat{\pi}_j \hat{p}_{ji} / \hat{\pi}_i$.
- 时间倒逆过程: $Y_t := X_{(T-t)-}$, $0 \leqslant t \leqslant T$.





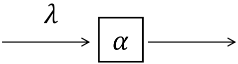
- 直观: $\{Y_t\}$ 的等待时间为q_i, 转移概率为$\hat{q}_{ij} = \hat{\pi}_j \hat{p}_{ji} / \hat{\pi}_i$.
- 命题2.5.1. $\{Y_t\}$ 是跳过程, 速率矩阵为$\tilde{\mathbf{Q}}$.
- 可逆分布(定义2.5.2): π 是不变分布, 且满足细致平衡条件:

$$\pi_i q_{ij} = \pi_j q_{ji}, \quad \forall i, j \in S.$$

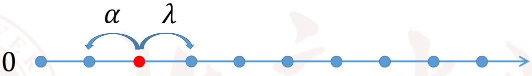
- 注: $\star\star \Rightarrow \pi \mathbf{Q} = 0$. 不能去掉$\star\star$/常返性假设, 反例2.6.1.
- 可逆过程: π 为可逆分布, $X_0 \sim \pi$.

例2.2.10 & 2.4.12 & 2.5.4. 排队系统.

- M/M/1:



- 建模: $S=\mathbb{Z}_+$. $\beta_i=\lambda$; $\delta_i=\alpha$.



- 嵌入链: $\mathbb{Z}_+$ 上带反射壁的RW.
  故, $\alpha<\lambda$: 非常返; $\lambda\leqslant\alpha$: 常返.

- M/M/s 与M/M/∞. 建模:

$$\delta_i=\begin{cases}\alpha i, & i\leqslant s,\\ \alpha s, & i>s,\end{cases} \qquad \delta_i=\alpha i.$$

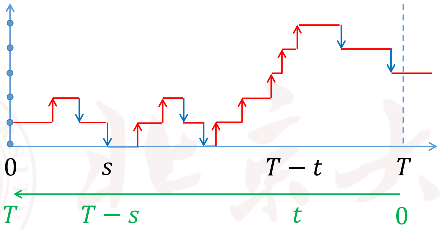
- 设$\lambda \leqslant \alpha$ (常返). 解$\pi\mathbf{Q} = 0$, 推出不变分布形如:

$$\pi_i = \pi_0 \frac{\beta_0 \cdots \beta_{i-1}}{\delta_1 \cdots \delta_i} = \pi_0 \left(\frac{\lambda}{\alpha}\right)^i, \quad i \geqslant 1.$$

- $\lambda = \alpha$: 零常返(不可归一化). $\lambda < \alpha$: 正常返(可归一化).
- 设$\lambda < \alpha$ (正常返). $\pi_0 = \frac{\alpha-\lambda}{\alpha}$. $X_0 \sim \pi$ (vs 几何分布).
- 平均长度: $EX_t = \frac{\lambda}{\alpha-\lambda}$, $X_t \sim \pi$.
- 服务员连续工作时间: $E_0\sigma_0 - E_0 S_1 = \frac{1}{q_0\pi_0} - \frac{1}{q_0} = \frac{1}{\mu-\lambda}$.
- 细致平衡条件成立, 故π 为可逆分布.

- $X_0 \sim \pi$, 则$\{X_t\}$ 是可逆过程. 因此

$$\Xi_T := \{S_1, \cdots, S_N\} \overset{d}{=} T - \Xi_T, \quad \Xi_T = \{T_1, \cdots, T_M\}$$



- 顾客离开的时刻Ξ 形成泊松流, 速率为λ.
- 进一步, 考虑第二个(串联)服务站.

- 转移速率:

$$q_{(i,j)(i+1,j)} = \lambda, \; q_{(i+1,j)(i,j+1)} = \alpha, \; q_{(i,j)(i,j-1)} = \beta.$$

- 不变分布: $\mu_{(i,j)} = (\frac{\lambda}{\alpha})^i \frac{\alpha-\lambda}{\alpha} \times (\frac{\lambda}{\beta})^j \frac{\beta-\lambda}{\beta}$.